# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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VOLUME 60

1953

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND SEATTLE, WASH.

# VIEWS AND APPROXIMATIONS ON DIFFERENTIAL EQUATIONS*

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**1. Introduction.** The English mathematician Andrew Russell Forsyth, 1858–1942, wrote sixteen books including his six-volume *Theory of differential equations* published from 1890 to 1906. In an obituary of Forsyth, E. H. Neville† (p. 237) said that the six-volume treatise is the most imposing single-handed work of any English mathematician. Forsyth believed that differential equations should be solved, and he set himself the task of collecting all effective methods for obtaining solutions of differential equations. Neville† (p. 242) said that Forsyth is more to blame than anyone else if to this day an applied mathematician always asks whether and how a differential equation can be solved instead of sometimes asking what the equation tells us about a function which satisfies it.

We should not ignore the necessity of knowing and teaching standard procedures for obtaining useful formulas for solutions of differential equations of important types. But we should hold the view that such procedures represent only one step, sometimes very useful and sometimes useless, in the process of learning about a function by deriving a differential equation which it satisfies and then squeezing information from the differential equation. Our attention in this paper is to be focused on methods of obtaining information from differential equations without solving the equations in any traditional sense. Sections 2, 3, and 4 indicate, by consideration of examples, methods independent of numerical calculations.

**2. A first-order equation.** To illustrate by an example the advantage of looking at a differential equation without solving it, we consider the amusing equation

$$y'(x) = A \sin x \sin y, \tag{2.1}$$

where $A$ is a positive constant. Simple theorems imply that when $a$ and $b$ are given constants there is one and only one function $y(x)$, defined over $-\infty < x < \infty$, for which $y(a) = b$ and (2.1) hold. Since $\sin x$ and $\sin y$ are alternately positive and negative over intervals of length $\pi$, we convert the whole $x, y$ plane into

* This is the first part of a lecture given June 26, 1951, by invitation of the program committee, at the joint meeting of the Mathematical Association of America and the American Society for Engineering Education held at Michigan State College. The remainder of the lecture dealt with numerical approximations.

† Neville, E. H. 1942, Andrew Russell Forsyth. (Obituary). Journal of the London Mathematical Society, vol. 17, pp. 237–256.

a checkerboard of squares having sides of length $\pi$ as in Figure 1. The graph of $y(x)$ has positive slope wherever it lies in a black (or shaded) square, negative slope wherever it lies in a white square, and zero slope wherever it intersects a boundary between black and white squares. It is immediately obvious from this fact, as it is from the equation (2.1) itself, that $y=n\pi$ is a solution of (2.1) for each integer $n$. Since no two graphs of solutions $y(x)$ can intersect unless they coincide, we obtain the very significant fact that if the graph of a solution $y(x)$ contains one point between two consecutive lines in the set $y=n\pi$,

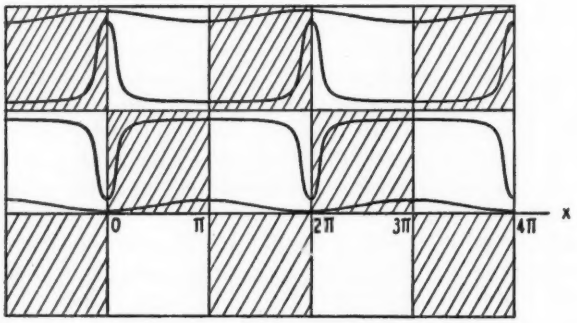


FIG. 1

then the whole graph lies between these lines. The symmetries in sin $x$ and sin $y$ imply that the lines $x=m\pi$, $m=1, 2, \cdots$, are lines of symmetry of graphs of solutions, and that solutions have period $2\pi$. Figure 1 shows the nature of graphs of some of the solutions of (2.1) when $A$ is a rather large positive constant.

The standard way to learn about the solutions of (2.1) is to separate the variables by dividing by sin $y$, integrate, and simplify. Depending on the formula one uses for the integral of the cosecant, we arrive at one or the other of the equivalent formulas

$$y = \pm 2 \tan^{-1} e^{c_1 - A\cos x} \tag{2.2}$$

$$y = \pm \cot^{-1} \sinh (c_2 + A \cos x). \tag{2.3}$$

These formal calculations should be accompanied by consideration of the possibility that sin $y$ may be 0, and of the manner in which the inverse trigonometric functions and the signs are to be used to isolate different solutions $y(x)$. In this case, information obtained by looking at the differential equation and the formulas obtained by solving the equation are both helpful. Each aids the other.

A more complicated equation such as

$$y' = \frac{A \sin x \sin y}{1 + x^2 + y^2} \tag{2.4}$$

resists solution, but consideration of the checkerboard and slopes yields fundamental ideas about its solutions. Perhaps the most important single fact about solutions of (2.4)is the fact that for each solution $\lim_{x\to\infty} y(x)$ must exist. If $y(x)$ satisfies (2.4), then $|y'(x)| < |A|x^{-2}$; hence, when $0<u<v$,

$$\begin{aligned} |y(v) - y(u)| &= \left| \int_u^v y'(x)dx \right| \leqq \int_u^v |y'(x)|\,dx \\ &< |A| \int_u^v x^{-2}dx = A\left(\frac{1}{u} - \frac{1}{v}\right). \end{aligned} \tag{2.5}$$

Hence, when $u$ and $v$ become infinite, $y(v)-y(u)\to 0$ and therefore $\lim_{x\to\infty} y(x)$ exists.

**3. Some views of the equation of the harmonic oscillator.** Let us suppose that a function $y(x)$ in which we are interested satisfies the conditions $y(0)=0$, $y'(0)=1$ and the differential equation

$$y''(x) + y(x) = 0. \tag{3.1}$$

This equation is linear, and we should know, or be able to discover very quickly, that each solution of (3.1) has the form $y=c_1\cos x+c_2\sin x$ and that the supplementary conditions determine $c_1$ and $c_2$ to make

$$y = \sin x. \tag{3.2}$$

By accident of our training, we know so many tables and properties of sin $x$ that we correctly consider the problem to be completely solved.

Suppose we were totally ignorant of the trigonometric solution of (3.1). We could seek power series solutions of (3.1) and find by standard procedures that

$$y(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \tag{3.3}$$

Of course we can use (3.3) to compute $y$ for any given value of $x$ and hence to make tables and graphs. But the behavior of $y(x)$ for large values of $x$, and in particular the existence of a strange number $\pi=3.141592653589793\cdots$ such that $y(x+2\pi)=y(x)$, is certainly not evident from (3.3). The fundamental fact is that one has learned very little when one has first reached (3.3). One who feels that differential equations should be solved and discarded may be partly disillusioned by discovering that the best way to learn about the function defined by (3.3) is to show that it satisfies (3.1) and then proceed as in the next paragraph.

Suppose another person, ignorant of the trigonometric solution, should write (3.1) in the form $y''(x) = -y(x)$ and try to sketch an approximation to the graph of $y(x)$. As $x$ starts at 0 and increases, the graph (Figure 2) starts at the origin and runs upward with slope beginning at 1 and (because $y''(x) = -y(x)$ is the rate of change of slope) decreasing at an increasing rate. Hence the slope must reach 0 for some value $x_1$ of $x$. As $x$ increases beyond $x_1$, the slope continues to decrease and the graph must intersect the $x$ axis at some point for which, say $x = x_2$. Because $y''(x)$ is a function of $y$ only, it can be concluded that the graph over the interval $0 \leqq x \leqq x_2$ is symmetric about the line $x = x_1$ and hence that $x_2 = 2x_1$ and $y'(x_2) = -1$. Because of this and the fact that $y''(x)$ is an odd function of $y$, it can be concluded that $y(x_2 + x) = -y(x)$ when $0 \leqq x \leqq x_2$. Thus when $x_4 = 2x_2 = 4x_1$ we have $y(x_4) = 0$ and $y'(x_4) = 1$. This and the fact that $y''(x)$ is a

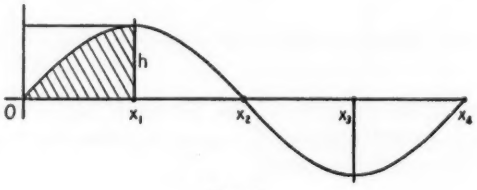


FIG. 2

function of $y$ only imply that the graph to the right of $x_4$ must be a copy of the graph to the right of 0. Thus we have an idea of the nature of the graph from 0 to $x_4$, and we know the fundamental fact that $y(x)$ is periodic, that is, $y(x + x_4) = y(x)$. If we multiply (3.1) by $2[y'(x)]$, we obtain

$$\frac{d}{dx}\{[y'(x)]^2 + [y(x)]^2\} = 0. \tag{3.4}$$

The quantity in braces is therefore a constant which must be 1 because $y(0) = 0$, $y'(0) = 1$. Therefore

$$[y'(x)]^2 + [y(x)]^2 = 1. \tag{3.5}$$

Since $y'(x_1) = 0$ and $y(x_1) > 0$, it follows that $y(x_1) = 1$. Again using (3.1), we find that

$$\int_0^{x_1} y(x)dx = -\int_0^{x_1} y''(x)dx = -y'(x_1) + y'(0) = 1. \tag{3.6}$$

If, for a rough first approximation, we assume that the area of the shaded region in Figure 1 is two-thirds of the area of the enclosing rectangle of length $x_1$ and height $h = 1$, we obtain $1 = (2/3) \cdot 1 \cdot x_1$ or $x_1 = 3/2$ and $x_2 = 3$. Thus it is not unreasonable to guess that $x_2$, normally called $\pi$, is about 3. Numerical methods

can be used to approximate $x_1$ and the function $y(x)$ over the interval $0 \leqq x \leqq x_1$. We now find ourselves developing the theory of the function $y(x)$, which is sin $x$, and $y'(x)$, which is cos $x$, with the aid of the differential equation (3.1), but we do not have time to continue the development here.

Let us now apply different points of view to a more complicated equation of the type $y''(x) = f(y)$ where $f(y)$ is an odd continuous increasing function of $y$, say

$$y''(x) = -y^3\sqrt{1 + y^2 + \cos^2 y} \tag{3.7}$$

where we assume as before the simple conditions $y(0) = 0$, $y'(0) = 1$. This equation is nonlinear. A power series expansion of $y(x)$ could not be obtained easily and could not provide much illumination even if one had it. An attempt to solve (3.7) by the standard device of multiplying by $2y'(x)$ and integrating leads, as it usually does, to the uninformative inverse of a function defined by an integral of a function of an integral not expressible in terms of elementary functions. It thus appears that we are not being successful in attempts to get information about $y(x)$ by solving (3.7). But if we will simply turn to the equation (3.7) itself and apply the same considerations involving slopes and rates of change of slope that we applied to (3.1), we see that $y(x)$ is periodic and that the graph over a period looks much like the graph in Figure 2, being somewhat straighter near its intersections with the $x$ axis. Methods of approximation can be used to approximate $x_1$, $x_2 = 2x_1$, $x_4 = 4x_1$, and the graph over a period. For the still more complicated equation

$$y''(x) = -y^3\sqrt{1 + x^2 + y^2 + \cos^2 y} \tag{3.8}$$

considerations of slopes and changes of slope will show that the symmetries and periodicities have disappeared and will give preliminary ideas about the manner in which $y(x)$ oscillates as $x$ increases.

**4. An eigenvalue problem.** For each real constant $\lambda$, let $y_\lambda(x)$ denote the real function $y(x)$ for which $y(-1) = 0$, $y'(-1) = 1$, and

$$y'' = -(\lambda - x^2)y. \tag{4.1}$$

Each solution of (4.1) which vanishes when $x = -1$ has the form $Cy_\lambda(x)$ where $C$ is a constant, and we focus our attention on the functions $y_\lambda(x)$. If $y_\lambda(1) = 0$, then $\lambda$ is an eigenvalue of a special eigenvalue problem, and $Cy_\lambda(x)$ is an eigenfunction belonging to $\lambda$ for each $C \neq 0$. For some purposes, it is better to write (4.1) in the form $-y'' + x^2y = \lambda y$, but this need not concern us here. However much theory one may know concerning this and related problems, it is of interest to see that qualitative information is readily obtained, from (4.1) itself, about all functions $y_\lambda(x)$ as well as about eigenvalues and eigenfunctions. If $\lambda \leqq 0$, then $y_\lambda'(x)$ increases from 1 and $y_\lambda(x)$ increases from 0 as $x$ increases from $-1$. Thus if $\lambda \leqq 0$, then $y_\lambda(1)$ cannot be 0 and $\lambda$ cannot be an eigenvalue. If $0 < \lambda < 1$, then $y_\lambda'(x)$ is increasing over $-1 < x < -\lambda^{1/2}$, decreasing over $-\lambda^{1/2}$

$<x<\lambda^{1/2}$, and increasing over $\lambda^{1/2}<x<1$, provided (as turns out to be the case) $y_\lambda(x)>0$ over $0<x<1$. The effect on $y_\lambda(x)$ of an increase in $\lambda$ is not immediately discernible from (4.1), because an increase in $\lambda$ forces $y_\lambda''(x)$ and hence $y_\lambda(x)$ to increase more rapidly at the start, and the resulting increase in $y$ in (4.1) tends, when $|x|<\lambda^{1/2}$, to make $y''$ smaller and to make $y'$ decrease more rapidly. To help us with our difficulties, we use a standard device. Supposing $\mu\neq\lambda$, we write

$$y_\lambda'' = -(\lambda - x^2)y_\lambda, \qquad y_\mu'' = -(\mu - x^2)y_\mu. \tag{4.2}$$

Multiplying these equations by $y_\mu$ and $-y_\lambda$ respectively and adding gives

$$y_\mu y_\lambda'' - y_\lambda y_\mu'' = (\mu - \lambda)y_\mu y_\lambda. \tag{4.3}$$

The left member of (4.3) is the derivative with respect to $x$ of $y_\mu(x)y_\lambda'(x) - y_\lambda(x)y_\mu'(x)$. Hence, since $y_\mu(-1)=y_\lambda(-1)=0$,

$$y_\mu(x)y_\lambda'(x) - y_\lambda(x)y_\mu'(x) = (\mu - \lambda)\int_{-1}^{x} y_\mu(t)y_\lambda(t)dt. \tag{4.4}$$

When $y_\mu(x)\neq 0$, we can divide by $[y_\mu(x)]^2$ to obtain

$$\frac{d}{dx}\frac{y_\lambda(x)}{y_\mu(x)} = \frac{\mu - \lambda}{[y_\mu(x)]^2}\int_{-1}^{x} y_\mu(t)y_\lambda(t)dt. \tag{4.5}$$

The two formulas (4.4) and (4.5) are excellent sources of information. If $\lambda<\mu$ and $y_\lambda(x)>0$ over an interval $-1<x<x_0$, then (4.5) and the fact that $\lim_{x\to -1}[y_\lambda(x)/y_\mu(x)]=1$ imply that $y_\lambda(x)/y_\mu(x)$ increases from 1 as $x$ increases over $-1<x<x_0$ and hence $y_\lambda(x)>y_\mu(x)$ over $-1<x<x_0$. All eigenvalues are greater than 1 since, as crude numerical calculations or the formula

$$y_1(x) = e^{-1/2}e^{-x^2/2}\int_{-1}^{x} e^{u^2}du \tag{4.6}$$

will show, $y_1(x)>0$ when $x>-1$; and, moreover, $y_\lambda(x)>y_1(x)$ when $0<\lambda<1$ and $x>-1$. If $\lambda\leqq 1$, then, in the interval $-1\leqq x\leqq 1$ where our interest lies, $y_\lambda'(x)$ is decreasing when $y_\lambda(x)$ is positive and increasing when $y_\lambda(x)$ is negative.

Using information derived above, and with repeated appeals to the fact involving rate of change of slope expressed in (4.1), it is quite easy to sketch schematically correct graphs of functions $y_\lambda(x)$ for many values of $\lambda$ greater than 1. It is very instructive to see how the number of oscillations of $y_\lambda(x)$ about the $x$ axis increases as $\lambda$ grows, and to see how it happens that, for roughly equally spaced values of $\lambda^2$, we have eigenvalues for which $y_\lambda(1)=0$. We cannot give an account of all useful facts made intuitively evident by these graphs. For example, it becomes very clear that values of $\lambda$ for which $y_\lambda'(1)=0$ (or, more generally, those for which $y_\lambda'(1)=ky_\lambda(1)$ where $k$ is a given constant) must interlace with the values of $\lambda$ for which $y_\lambda(1)=0$.

# A DIFFERENT APPROACH TO THE NON-EUCLIDEAN GEOMETRIES

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The starting point for our approach to the trigonometry of the non-Euclidean geometries is a simple question. What is the nature of a 3-space in which the ordinary spherical trigonometry holds? By the latter we mean the trigonometry of a trihedral angle imbedded in our space. To formulate our problem more precisely we put it on an axiomatic basis. We take Hilbert's axioms of incidence, order, and congruence [1, pp. 25–31] or any equivalent set of axioms. Since we want to leave out the axioms of parallelism later on, we need an axiom of continuity independent of any assumption on parallel lines. We have a choice of Dedekind's axiom [3, p. 37] or the axioms of Eudoxus and Cantor [1, pp. 41–45]. Upon adding the Euclidean axiom of parallels [1, p. 55] the system of axioms for the Euclidean geometry is complete. The spherical trigonometry can now be derived in the usual fashion. If we omit the axiom of parallelism, the other axioms together with the spherical trigonometry still constitute a consistent geometric system. In this paper we investigate the geometries to which this system gives rise.

We state first some geometric facts which are true in the system we are considering and which will be referred to later. The dihedral angle between two planes is measured by the angle between two lines, one in each plane, perpendicular to the line of intersection at a common point. This measure is independent of the point chosen on the line of intersection [5, p. 42]. Furthermore there are three theorems of solid geometry which can be shown without reference to parallels [5, p. 43]: (I) If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to every line through this point in the plane of the two lines. (II) If a line is perpendicular to a given plane, every plane containing this line is perpendicular to the given plane. (III) Any line which lies in one face of a right dihedral angle and is perpendicular to the edge is perpendicular to the other face.

We now construct a familiar configuration. Let $ABC$ be a right triangle having the right angle at $C$. Set $BC=a$, $CA=b$, $AB=c$. Consider a segment $VA$ of arbitrary length perpendicular to $AB$ and $AC$. The line $VA$ is then perpendicular to the plane $ABC$ (I). Also plane $ABC$ is perpendicular to plane $VCA$ (II). Since $BC$ is perpendicular to $CA$ it follows that $BC$ is perpendicular to plane $VCA$ (III). Especially $BC$ is perpendicular to $VC$. Moreover plane $VBC$ is perpendicular to plane $VCA$ (II). Thus $V-ABC$ is a trihedral angle whose dihedral angle $VC$ is right. The dihedral angle $VA$ is measured by the angle $A$ of triangle $ABC$. All four triangles in our configuration are right triangles. We finally set $VB=p$, $VC=q$ and for the face angles of the trihedral angle $BVC=a'$, $CVA=b'$, $AVB=c'$. The trihedral angle $V-ABC$ will be used throughout the following discussion.

The ordinary congruence theorems are valid in our geometric system [1, pp. 33–38]. Hence two right triangles are congruent if any two sides of one are

respectively equal to the corresponding sides of the other. In other words, if two sides are given the other parts of the right triangle are completely determined. Now the axioms of continuity allow us to measure distances and angles. More exactly we can establish a one-to-one correspondence between certain sets of real numbers and distances or angles respectively [1, pp. 42–47]. If then the measures of two sides of a right triangle are given, these sides are determined. So are the remaining parts of the triangle and hence their measures. Especially the measure of an angle or a given function of it will be a certain function of the measures of the given sides.

We apply the last conclusion to the right triangle $ABC$ and set $\sin A = S(a, c)$ where $S$ stands for a certain unknown function. Similarly $\sin a' = S(a, p)$ and $\sin c' = S(c, p)$. From spherical trigonometry we know that $\sin A = \sin a'/\sin c'$. Hence we have $S(a, c) = S(a, p)/S(c, p)$. We can show that in this relation $p$ remains arbitrary. It must be greater than $c$, however [1, p. 38]. After the triangle $ABC$ has been drawn the direction of $VA$ is fixed. The length of $VA$ is found from triangle $VAB$ once $p$ is chosen. Thus our trihedral angle can be constructed selecting $p$ arbitrarily. Because of this we can put $S(a, p) = s(a)$ for some value of $p$ with $s$ representing an unknown function. Hence

$$\sin A = \frac{s(a)}{s(c)}. \tag{1}$$

With arguments analogous to the preceding ones we set $\tan A = T(a, b)$. Using $\tan A = \tan a'/\sin b'$ and (1) we obtain $T(a, b) = T(a, q)s(q)/s(b)$. Again the trihedral angle can be completed selecting $q$ at will. We therefore set $T(a, q)s(q) = t(a)$. Then

$$\tan A = \frac{t(a)}{s(b)}. \tag{2}$$

Making use of $\cos A = \tan b'/\tan c'$ and (2) we find

$$\cos A = \frac{t(b)}{t(c)}. \tag{3}$$

We now equate the right side of (3) to the expression for $\cos A$ derived by dividing (1) and (2). Hence $s(a)s(b)/t(a)t(b) = s(c)/t(c)$. Upon introducing a new unknown function $f$ by means of the defining identity

$$f(a) = \frac{s(a)}{t(a)} \tag{4}$$

this becomes

$$f(a)f(b) = f(c). \tag{5}$$

We note as a simple consequence of (1), (3), (4), (5)

$$\frac{\cos A}{\sin B} = f(a). \tag{6}$$

In order to find one more relation between our unknown functions we substitute (1) and (3) in the identity $\sin^2 A+\cos^2 A=1$. With the aid of (4) and (5) we can write $s^2(a)+f^2(a)s^2(b)=s^2(c)$. We equate the left side to the expression we obtain from it interchanging $a$ and $b$. Thus

$$\frac{f^2(a)-1}{s^2(a)} = \frac{f^2(b)-1}{s^2(b)} = e. \tag{7}$$

Here we have set the common ratio equal to $e$. The quantities $a$ and $b$ are the measures of the legs of a right triangle. After a value has been chosen for $a$ the value of $b$ can still be selected arbitrarily. Consequently $e$ is a universal constant for the relation (7) between the unknown functions $f$ and $s$. We also see that in (1), (2), (3), (4) only ratios of $s$ and $t$ are involved. This makes it permissible to consider $s$ multiplied by a factor such that $|e|=1$. Since only the square of $s$ occurs in (7) we can also assume $s$ positive. According as the numerators in (7) are positive or negative $e$ will be 1 or $-1$. There is a third possibility too, namely $e=0$. In this case $s$ could not be normalized by the above procedure. However, no such necessity will arise.

As the last step in our deduction we establish a functional equation for $f$. For this purpose we draw in triangle $ABC$ the altitude $CD=z$ which divides the hypotenuse into two segments $AD=x$ and $BD=y$. We set angle $ACD=\theta$ and angle $BCD=\phi$. These angles being complementary we have $\tan\theta\tan\phi=1$. On using (2) for expressing this fact we see that

$$t(x)t(y) = s^2(z). \tag{8}$$

Applying (5), (7), (8) in succession we conclude that

$$\begin{aligned} f(x+y) &= f(a)f(b) = f^2(z)f(x)f(y) = [1+es^2(z)]f(x)f(y) \\ &= [1+et(x)t(y)]f(x)f(y), \end{aligned}$$

or in accordance with (4)

$$f(x+y) = f(x)f(y) + es(x)s(y). \tag{9}$$

For a special case we make $ABC$ an isosceles right triangle and $x$ will be equal to $y$ in (9). It then follows from (7) that

$$f(2x) + 1 = 2f^2(x). \tag{10}$$

We have arrived at the functional equations (9) and (10) by means of the right triangle $ABC$. But the equations are meaningful only if such a triangle exists for arbitrarily chosen $x$ and $y$. For a proof let our notations apply to oblique triangles $ABC$. Because of (6) $\cos\theta/\sin A=f(x)$. Now in our geometric system the angles $\theta$ and $A$ are necessarily acute angles [1, p. 48]. Hence if we

hold $x$ constant $\theta$ is a continuous decreasing function of $A$. On the other hand it is geometrically obvious that $A$ is a continuous increasing function of $z$. Thus $\theta$ is a continuous decreasing function of $z$ for a fixed $x$. More generally, $C=\theta+\phi$ will be a continuous decreasing function of $z$ for given values of $x$ and $y$. Assume for definiteness that $x$ and $y$ are given such that $x$ is greater than $y$. In this case $\theta$ is greater than $\phi$. If we make $\theta$ equal to one half of a right angle, the right triangle $ACD$ is determined by $x$ and $\theta$ [1, p. 38]. In this way we find a value of $z$ corresponding to an angle $C$ less than a right angle. Again, make $\phi$ equal to one half of a right angle. This time we find a value of $z$ corresponding to an angle $C$ greater than a right angle. Since $C$ is a continuous and decreasing function of $z$, there will be exactly one $z$ between the two values considered for which $C$ is a right angle. If $x$ and $y$ are equal the two values of $z$ simply coincide. This completes the proof.

We return now to the right triangle $ABC$. Using (3) we find two expressions for $\cos B$ in triangles $BCD$ and $ABC$. With a manipulation similar to the preceding we then see that

$$f(c-y)=f(c)f(y)-\varepsilon s(c)s(y). \tag{11}$$

As before $c$ and $y$ may take on arbitrary values with the only restriction that $c$ is greater than $y$. Addition of (9) and (11) with an obvious change of letters leads to

$$f(x+y)+f(x-y)=2f(x)f(y). \tag{12}$$

Two conditions are usually imposed on the $f$-function in the functional equations (10) and (12). We show that our unknown function satisfies these conditions. First, $f$ has to be continuous. We know already that $A$ and $\theta$ are continuous functions of $z$. Thus in the relation $\cos A/\sin\theta=f(z)$ due to (6) $f(z)$ is a continuous function of $z$. Second, $f(x)$ in (10) must be positive. This becomes obvious from $\cos\theta/\sin A=f(x)$, where $\theta$ is one half of a right angle for the special case contemplated in (10). Under the stated conditions the solutions of our functional equations are [2, p. 52]

$$\cosh\frac{x}{k},\quad \cos\frac{x}{k},\quad 1$$

with an arbitrary constant $k$. This constant can be made equal to unity for a suitable choice of the unit of length. The values of $\varepsilon$ in (7) corresponding to the above solutions are $1, -1, 0$. For the first two solutions the functions $s$ and $t$ are found from (7) and (4). They are $\sinh a$ and $\tanh a$, $\sin a$ and $\tan a$, respectively. As a glance at (1), (2), (3), (5), (6) will show, we are dealing with formulas of hyperbolic and elliptic trigonometry. These formulas will more than suffice to build up these trigonometries. The elliptic geometry, however, does not satisfy all the axioms we had to start with. Yet it appears as a solution of our original problem. The explanation is this. Our axioms are true for the elliptic

case, if we restrict ourselves to a limited region of our space. In other words, all the segments involved should not exceed a certain length [4, pp. 6, 28, 29]. Clearly all the triangles considered can be made small enough and all conclusions will apply to elliptic geometry. The third solution of our functional equations does not allow us to find $s$ and $t$ in the available equations. But we can infer from (6) that $A$ and $B$ are complementary angles in this case. This is known to be equivalent to the Euclidean axiom of parallels [3, p. 46]. Thus the ordinary Euclidean geometry constitutes the third solution of our original problem. It was clear from the beginning of course that it had to be among the solutions.

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## THE TREATMENT OF BOUNDARY PROBLEMS BY MATRIX METHODS

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**1. Introduction.** Various methods leading to the approximate solution of systems of ordinary linear differential equations with boundary conditions are available; some of the most useful ones are Taylor series expansions, collocation, Galerkin's method, least squares solution, *etc.*

The purpose of this article is two-fold. First it is intended to acquaint the reader with the least squares approach, and, secondly, it is shown how the use of matrix procedures may be employed advantageously to give a more compact description of this least squares method. This method is essentially an application of the Rayleigh-Ritz principle in that the problem to be solved is replaced by a related variational problem for which approximate solutions may be found with relative ease [1].

As a by-product it is interesting to note that computations carried out for a number of cases seem to indicate a superiority of the least squares method over the ones mentioned above.

It is a pleasure to acknowledge that certain suggestions made by B. Garfinkel helped materially in the preparation of this paper.

**2. The least squares method.** Let us consider the linear differential equation of order $n$

$$L(D)y \equiv \sum_{k=0}^{n} p_k(x)D^{n-k}y = r(x), \tag{1}$$

where $D^k \equiv d^k/dx^k$, $k=0, 1, 2, \cdots, n$, $D^0y=y$, and the $p_k$, $r$ are continuous functions almost everywhere in some interval $I$: $x_0 \leqq x < x_1$. We are interested in the solution $y(x)$ of (1) which satisfies boundary conditions that assure the unique existence of this solution. Suppose these conditions are of the form

$$\Lambda(D)y(a) = \rho; \tag{2}$$

here

$$a \equiv \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_p \end{bmatrix}$$

is a column of prescribed positions $a_k \epsilon I$, $k=1, 2, \cdots, p$, and

$$y(a) \equiv \begin{bmatrix} y(a_1) \\ y(a_2) \\ \cdot \\ \cdot \\ \cdot \\ y(a_p) \end{bmatrix}.$$

Further, $\Lambda(D)$ is a matrix of $n$ rows and $p$ columns, and its elements are polynomials in $D$ of degree not exceeding $n-1$. Finally, $\rho$ is a column of $n$ constants, not all zero.

To obtain an approximate solution of the $p$-point boundary problem of the type described above we define a sequence $y_s(x)$, $s=1, 2, \cdots$ of approximations to the solutions in the following manner:

$$y_s(x) = \sum_{k=0}^{s} Y_k(x)c_k; \tag{3}$$

here the $Y_k$ are arbitrary functions of class $C^{n-1}$, with $Y_0$ satisfying the boundary conditions (2), the $Y_k$ for $k \geqq 1$ satisfying the associated homogeneous boundary conditions, and $c_0=1$, while the $c_k$, $k \geqq 1$, are unknown constants to be properly adjusted so as to make $y_s$ approximate solutions of (1). By defining

$$Y \equiv \begin{bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ Y_s \end{bmatrix} = (Y_i)$$

we may write (3) in the form $y_s = Y_0 + Y^T c$, and express the homogeneous bound-

ary conditions as $\Lambda(D)Y^T(a)=0$, with $Y^T$ denoting the transpose of $Y$: $Y^T \equiv (Y_1 Y_2 \cdots Y_s) = (Y_j)$.

While now the $y_s$ clearly satisfy the conditions (2):

$$\Lambda(D)y_s(a) = \Lambda(D)Y_0(a) + \Lambda(D)Y^T(a)c = \rho,$$

they will not, in general, fulfill the differential equation (1), so that the function

$$\epsilon(x) = L(D)y_s - r \tag{4}$$

will not vanish everywhere in $I$.

A variety of methods may now be employed in an effort to make $\epsilon(x)$ small in $I$ [2]. What is proposed here is to achieve this by the method of least squares, minimizing, namely, the function

$$U(c) = \int_{x_0}^{x_1} \epsilon^T(t)\epsilon(t)dt. \tag{5}$$

By (4) and (3)

$$\begin{aligned} \epsilon &= (LY^T)c - (r - LY_0) \\ &= Ac - b, \end{aligned}$$

if we put $A(x)=LY^T(x)$, $b(x)=r(x)-LY_0(x)$. It follows that

$$U(c) = c^T\left(\int A^T A dt\right)c - 2c^T\left(\int A^T b dt\right) + \int b^T b dt. \tag{6}$$

For a minimum of $U(c)$ it is necessary that

$$\nabla U \equiv \begin{bmatrix} \partial U/\partial c_1 \\ \vdots \\ \partial U/\partial c_s \end{bmatrix} = 0.$$

Now for a quadratic form $c^TBc$

$$\nabla(c^TBc) = (B + B^T)c.$$

Since in our case $B=\int A^TA dt$ is symmetrical,

$$\nabla c^T\left(\int A^T A dt\right)c = 2\left(\int A^T A dt\right)c.$$

For a linear form $c^TE$

$$\nabla(c^TE) = E.$$

Consequently we obtain the normal equations

$$\left(\int A^T A dt\right)c = \int A^T b dt,$$

*i.e.*

$$\left(\int_{x_0}^{x_1} LY(t)\cdot LY^T(t)dt\right)c = \int_{x_0}^{x_1} LY(t)[r(t) - LY_0(t)]dt. \tag{7}$$

The determination of the vector $c$ is thus reduced to the solution of the system (7) of linear equations with the symmetrical matrix $M=\int LY\cdot LY^T dt$.

Written out in more detail equations (7) look like this:

$$\int_{x_0}^{x_1}\begin{pmatrix} (LY_1)^2 & LY_1\cdot LY_2 & LY_1\cdot LY_3 & \cdot \\ \cdot & (LY_2)^2 & LY_2\cdot LY_3 & \cdot \\ \cdot & \cdot & (LY_3)^2 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} dt\cdot c = \int_{x_0}^{x_1}\begin{pmatrix} LY_1(r - LY_0) \\ LY_2(r - LY_0) \\ LY_3(r - LY_0) \\ \cdot \end{pmatrix} dt.$$

The case $s+1$ thus differs from the case $s$ only in that the terms due to $LY_{s+1}$ are inserted, without, however, changing the previous terms.

**3. Example.** Suppose we want to solve the differential equation

$$[D^2 - (1 - 4/x)]y = 2\sinh x \tag{8}$$

in the interval $0\leqq x\leqq 1$, subjecting the solution $y(x)$ to the boundary conditions

$$\begin{pmatrix} 1 & -D+2 \\ eD & -D+1 \end{pmatrix} y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}e \\ 0 \end{pmatrix}. \tag{9}$$

Taking $Y_0=a_0x+b_0x^2$, we find, by (9) that

$$a_0 = -\tfrac{1}{2}e, \qquad b_0 = -\tfrac{1}{2}.$$

Next we assume the $Y_i$, $i=1, 2, \cdots, s$ to be of the form

$$Y_i = a_ix^i + b_ix^{i+1} + x^{i+2}.$$

The homogeneous boundary conditions then show that $a_i=1$, $b_1=e-2$, $b_i=-2$ for $i\geqq 2$, so that

$$\begin{aligned} Y_1 &= x[1 + (e - 2)x + x^2], \\ Y_i &= x^i(1 - x)^2, \qquad i \geqq 2. \end{aligned}$$

It follows that

$$\begin{aligned} LY = {} & i(i-1)x^{i-2} + [i(i+1)b_i + 4]x^{i-1} \\ & + [(i+1)(i+2) - (1 - 4b_i)]x^i + (4 - b_i)x^{i+1} - x^{i+2}, \end{aligned}$$

and $LY^T$ similarly, with the column index $j$ replacing the row index $i$. Consesequently,

$$LY\cdot LY^T = \sum_{k=0}^{8} d_k x^{i+j+k-4},$$

with $d_k$ denoting certain constants that depend on $i, j, b_i, b_j$; thus:

$$d_0 = i \cdot j(i-1)(j-1),$$
$$d_1 = j(j-1)[i(i+1)b_i + 4] + i(i-1)[j(j+1)b_j + 4], \cdots,$$
$$d_7 = b_i + b_j - 8,$$
$$d_8 = 1.$$

The elements $m_{ij}$ of $M$ are now easily obtained in the form

$$m_{ij} = \sum_{k=0}^{8} d_k/(i+j+k-3);$$

the result is, for $s=3$,

$$M = \begin{Bmatrix} 113.169252 & 2.82366572 & 1.22204903 \\ \cdot & .696825397 & .350000000 \\ \cdot & \cdot & .298845599 \end{Bmatrix}.$$

Having calculated the matrix $M$ we next proceed to compute the right-hand side $\int_{x_0}^{x_1} LY(r-Y_0)dt$ of (7); for our example we find

$$\begin{Bmatrix} 39.5069050 \\ .950329464 \\ .414999435 \end{Bmatrix}.$$

The components of $c$ found as the solution of (7) are shown in the first column of Table 1, accurate to six decimals. The second column of this table contains the values of the $c_i$ as found by Galerkin's Method:

*Table 1. Comparison of Solutions. Vector c*

| $c$ | Least Squares | | Galerkin's Method | | Exact Solution |
|---|---|---|---|---|---|
| | $s=3$ | $s=4$ | $s=3$ | $s=4$ | |
| $c_1$ | .350607 | .350606 | .350598 | .350601 | .350606 |
| $c_2$ | −.083317 | −.084951 | −.082594 | −.083449 | −.085168 |
| $c_3$ | .052537 | .060539 | .052006 | .055174 | .062392 |
| $c_4$ | | −.007879 | | −.002992 | −.012271 |

$$\left(\int_{x_0}^{x_1} Y \cdot LY^T dt\right) c = \int_{x_0}^{x_1} Y(r - LY_0)dt. \tag{10}$$

The last column exhibits the exact values of the $c_i$; the latter are easily calculated by expanding the exact solution of (8) and (9):

$$y(x) = (xe^x/6) - (x^2e^{-x}/2), \tag{11}$$

and comparing the coefficients of the successive powers of $x$ with the coefficients resulting from the approximate solution $y_s$, as defined by (3).

A comparison of these solutions indicates the superiority of the least squares method, the convergence to the exact values proceeding faster than in Galerkin's Method.

This superiority becomes even more apparent when the solutions are compared throughout the interval of integration, as shown in Table 2.

*Table 2. Comparison of Solutions. Function y*

| $x$ | 0 | .2 | .4 | .6 | .8 | 1.0 |
|---|---|---|---|---|---|---|
| Least Squares | | | | | | |
| $s=3$ | 0 | .024348 | .045817 | .083409 | .152962 | .269109 |
| $s=4$ | 0 | .024339 | .045828 | .083427 | .152954 | .269107 |
| Galerkin | | | | | | |
| $s=3$ | 0 | .024361 | .045835 | .083422 | .152953 | .269083 |
| $s=4$ | 0 | .024353 | .045832 | .083423 | .152953 | .269092 |
| Exact Solution | 0 | .024339 | .045829 | .083426 | .152954 | .269107 |

The square of the residuals, in the least squares procedure, drops from $727 \cdot 10^{-6}$, for $s=3$, to $2 \cdot 10^{-6}$, for $s=4$; in the Galerkin Method the corresponding numbers are $1113 \cdot 10^{-6}$ and $440 \cdot 10^{-6}$.

**4. Systems of equations.** The method described above may be applied, with suitable modifications, to systems of differential equations

$$f(D)y = r, \tag{12}$$

whose solutions are subject to boundary conditions

$$\phi(D)y(a) = \rho. \tag{13}$$

In (12), $f(D)$ is an $m \cdot m$ matrix, $m \geqq 1$, with elements that are polynomials in $D$ of degree $n \geqq 1$, $y$ is a vector of functions $y_1, y_2, \cdots, y_m$. In (13), $a$ again denotes a column of prescribed points $a_1, a_2, \cdots, a_p$, $a_k \epsilon I$. $\phi(D)$ is an $n \cdot ms$ matrix whose elements are polynomials in $D$ of degree not exceeding $n-1$, and $y(a)$ is to be formed as follows:

$$y(a) = \begin{bmatrix} y(a_1) \\ \cdot \\ \cdot \\ y(a_p) \end{bmatrix}, \quad y(a_k) = \begin{bmatrix} y_1(a_k) \\ \cdot \\ \cdot \\ y_m(a_k) \end{bmatrix}, \qquad k = 1, 2, \cdots, p.$$

As before, the $s$th approximation to the solution may be defined in the form

$$y_s(x) = Y_0(x) + Y^T(x)c. \tag{14}$$

Here

$$y_s = \begin{pmatrix} y_{1s} \\ \cdot \\ \cdot \\ y_{ms} \end{pmatrix}, \qquad Y_0 = \begin{pmatrix} Y_{10} \\ \cdot \\ \cdot \\ Y_{m0} \end{pmatrix},$$

and the independent $Y_{i0}$ must be chosen such that

$$\phi(D)Y_0(a) = \rho; \tag{15}$$

further, the $m \cdot ms$ matrix $Y^T$ must be constructed in the following manner: Let

$$Y_i = \begin{pmatrix} Y_{i1} \\ \cdot \\ \cdot \\ Y_{is} \end{pmatrix}, \qquad i = 1, 2, \cdots, m,$$

have independent components $Y_{ij}(x)$, and let

$$Y = \begin{pmatrix} Y_1 & 0 & \cdots & 0 \\ 0 & Y_2 & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & 0 \\ 0 & \cdots & 0 & Y_m \end{pmatrix} \tag{16}$$

denote the diagonal matrix of $ms$ rows and $m$ columns. Then

$$Y^T = \begin{pmatrix} \overline{Y}_1 & & & \\ & \overline{Y}_2 & & \\ & & \cdot & \\ & & & \overline{Y}_m \end{pmatrix},$$

with $\overline{Y}_1 = (Y_{i1}, Y_{i2}, \cdots, Y_{is})$, $i = 1, 2, \cdots, m$.

The $Y_{ij}$ must be selected in such a manner that $Y^T$ satisfies the $n$ homogeneous boundary conditions

$$\phi(D)Y^T(a) = \phi(D) \begin{vmatrix} \overline{Y}_1(a_1) & \cdots & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \cdots & \overline{Y}_m(a_1) \\ \cdot & \cdot & \cdot \\ \overline{Y}_1(a_p) & \cdots & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & \cdots & \overline{Y}_m(a_p) \end{vmatrix} = 0. \tag{17}$$

This prescription will again assure the satisfaction of the conditions (13) by the approximating functions $y_s$.

The vector $c$, in (14), is composed of $m$ vectors $c^{(i)}$ each having $s$ components $c_j^{(i)}$:

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$$c = \begin{vmatrix} c^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ c^{(m)} \end{vmatrix}, \qquad c^{(i)} = \begin{vmatrix} c_1^{(i)} \\ \cdot \\ \cdot \\ \cdot \\ c_s^{(i)} \end{vmatrix}, \qquad i = 1, 2, \cdots, m.$$

Minimizing the residuals $\epsilon = \phi(D)y_s - r$ by the method of least squares now leads to the normal equations

$$\left(\int_{x_0}^{x_1} fY \cdot fY^T dt\right) c = \int_{x_0}^{x_1} fY(r - fY_0)dt. \tag{18}$$

Care must be exercised to carry out the required matrix operation in the proper manner. Thus

$$fY = \begin{vmatrix} f_{11} \cdots f_{1m} \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ f_{m1} \cdots f_{mm} \end{vmatrix} \begin{vmatrix} Y_1 & & \\ & \cdot & \\ & & Y_m \end{vmatrix} = \begin{vmatrix} f_{11}Y_1 \cdots f_{1m}Y_m \\ \cdot \\ \cdot \\ f_{m1}Y_1 \cdots f_{mm}Y_m \end{vmatrix},$$

and similarly for $f\overline{Y}$. Since $fY$ is an $ms \cdot m$ matrix, $f\overline{Y}$ an $m \cdot ms$ matrix, $M = \int_{x_0}^{x_1} fYf Y^T dt$ is a symmetric matrix of $ms$ rows.

**5. Example of a system.** Let us apply this method to the system

$$\begin{pmatrix} D^2 & x^2D + 3x \\ 2 & x^3D^2 + x^3 - 2x \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{19}$$

with the boundary conditions

$$\begin{pmatrix} D & -1 & 0 & 0 \\ 1 & 0 & -2D/\pi & 0 \end{pmatrix} \begin{vmatrix} y_1(0) \\ y_2(0) \\ y_1(\pi/2) \\ y_2(\pi/2) \end{vmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{20}$$

Here $m = n = p = 2$. Suppose we take

$$Y_{10} = x - (x^3/2) + bx^5, \qquad Y_{20} = 1 - (x^2/6) + (x^4/120),$$

with $b$ to be determined such that (20) is fulfilled. We find

$$b = 2(3\pi^2 - 4\pi - 8)/5\pi^4.$$

Next we choose $Y_1$, $Y_2$ to be of the form

$$Y_{1j} = x^{j+1} + b_j x^{j+2}$$
$$Y_{2j} = x^j, \qquad \text{for } j = 1, 2, 3, 4,$$

taking $s = 4$. The $b_j$ must be computed from conditions (17); these lead to

$$b_j = -(2/\pi)(j + 1)/(j + 2).$$

It is seen that the normal equations (18), for the case (19), (20), have the solution

$$c^{(1)} = \begin{vmatrix} -0.00107 \\ 0.00670 \\ -0.01361 \\ 0.01500 \end{vmatrix}, \qquad c^{(2)} = \begin{vmatrix} 0.00593 \\ -0.01967 \\ 0.01859 \\ -0.01078 \end{vmatrix}.$$

The exact values are $c_4^{(1)} = 0.00453$, all other $c_j^{(i)} = 0$, $i = 1, 2$; $j = 1, \cdots, 4$, as may be verified by expanding the exact solution

$$y(x) = \begin{pmatrix} x \cos x \\ (1/x) \sin x \end{pmatrix}$$

in Taylor series about $x = 0$. Table 3 contains again a comparison of the approximating function with the exact solution, in the basic interval $(0, \pi/2)$.

*Table 3. Comparison of Solutions*

| | 0 | $\pi/16$ | $\pi/8$ | $3\pi/16$ | $\pi/4$ | $3\pi/8$ | $\pi/2$ |
|---|---|---|---|---|---|---|---|
| Exact Solution | 0 | .1926 | .3628 | .4898 | 0.5554 | 0.4508 | 0 |
| | 1 | .9936 | .9745 | .9432 | 0.9003 | 0.7842 | 0.6366 |
| Approximate Solution | 0 | .1926 | .3628 | .4901 | 0.5572 | 0.4724 | 0.1205 |
| | 1 | .9941 | .9747 | .9423 | 0.8978 | 0.7740 | 0.6067 |

**References**

1. For a description of this principle see, for example, R. Courant, Variational Methods for the Solution of Problems of Equilibrium and Vibrations, Bulletin, A. M. S., vol. 49, 1943.

2. Such methods are the Collocation Method, Galerkin's Method, etc., described in Frazer, Duncan, Collar, Elementary Matrices.

## A NEW METHOD OF EVALUATING $\zeta(2n)$

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All the extant determinations of $\sum_{\nu=1}^{\infty} (1/\nu^{2n})$ turn on arguments quite remote from the series itself, which generally makes its adventitious appearance in the next to the last line of the proof. For instance, Riemann's functional equation is obtained by transforming the series into a definite integral, which is in turn transformed into a contour integral, which is transformed back into a series (of residue terms) which is luckily of the same form as the original one; or,

alternatively, the function $1/(e^z-1)-1/z$ is expanded about its symmetrically-paired poles, the summand is written as an infinite series and the order of summation is inverted, yielding a power-series, whose coefficients, which turn out to be $\zeta(2n)$, are equated with those of the generating function, these being assumed to be known (or, at least, knowable) constants.

It is consequently hoped that the following approach, which is wholly elementary and deals only with manipulations of the defining series, may be of intrinsic as well as didactic interest. Except for digressions in (8) and (12), where the passage to the limit is justified, the argument moves ahead under its own momentum, once the initial steps are taken. The convolutions obtained (Theorems I and II) seem never to have been explicitly formulated before, and Theorem III is apparently entirely new.

THEOREM I:

$$\zeta(2)\zeta(2n-2)+\zeta(4)\zeta(2n-4)+\cdots+\zeta(2n-2)\zeta(2)=(n+\tfrac{1}{2})\zeta(2n) \qquad (n=2,3,4,\cdots).$$

The left-hand side, written out at length, is the limit, as $N\to\infty$, of

$$\sum_{\nu=1}^{N}\sum_{\mu=1}^{N}\left\{\frac{1}{\mu^2}\cdot\frac{1}{\nu^{2n-2}}+\frac{1}{\mu^4}\cdot\frac{1}{\nu^{2n-4}}+\cdots+\frac{1}{\mu^{2n-2}}\cdot\frac{1}{\nu^2}\right\} \tag{1}$$

($N$ a positive integer). Summing the geometric series within braces, and taking note of the exceptional case $\mu=\nu$, this becomes

$$\sum_{\nu}\left\{\sum_{\mu}{}'\frac{\nu^{2-2n}-\mu^{2-2n}}{\mu^2-\nu^2}+(n-1)\frac{1}{\nu^{2n}}\right\}. \tag{2}$$

Throughout the present discussion, all sums run from 1 to $N$, unless otherwise indicated, and an accent on an inner $\sum$ indicates that the index (in this case, $\mu$) does not take on the value of the index of the outer sum ($\nu$, here). Ignoring the term on the far right, (2) is equal to

$$\sum_{\nu}\sum_{\mu}{}'\frac{\nu^{2-2n}}{\mu^2-\nu^2}+\sum_{\nu}\sum_{\mu}{}'\frac{\mu^{2-2n}}{\nu^2-\mu^2}. \tag{3}$$

Inverting the order of summation in the second double sum, and noting that the condition $\mu\neq\nu$ is the same as the condition $\nu\neq\mu$, (3) may be written as

$$\sum_{\nu}\frac{1}{\nu^{2n-2}}\sum_{\mu}{}'\frac{1}{\mu^2-\nu^2}+\sum_{\mu}\frac{1}{\mu^{2n-2}}\sum_{\nu}{}'\frac{1}{\nu^2-\mu^2}=2\sum_{\nu}\frac{1}{\nu^{2n-2}}\sum_{\mu}{}'\frac{1}{\mu^2-\nu^2}, \tag{4}$$

the latter form arising out of an interchange of the dummy indices in the second sum. Combining (1)–(4), we find that

$$\sum_{\nu}\sum_{\mu}\left\{\frac{1}{\mu^2}\cdot\frac{1}{\nu^{2n-2}}+\cdots+\frac{1}{\mu^{2n-2}}\cdot\frac{1}{\nu^2}\right\} = (n-1)\sum_{\nu}\frac{1}{\nu^{2n}}+2\sum_{\nu}\frac{1}{\nu^{2n-2}}\sum_{\mu}{}'\frac{1}{\mu^2-\nu^2}. \tag{5}$$

Now,

$$\begin{aligned}
2\nu\sum_{\mu}{}'\frac{1}{\mu^2-\nu^2} &= \sum_{\mu}{}'\frac{1}{\mu-\nu}-\sum_{\mu}{}'\frac{1}{\mu+\nu}\\
&= \sum_{\mu=1}^{\nu-1}\frac{1}{\mu-\nu}+\sum_{\mu=\nu+1}^{N}\frac{1}{\mu-\nu}-\sum_{\mu=1}^{N}\frac{1}{\mu+\nu}+\frac{1}{2\nu}\\
&= -\sum_{\mu=1}^{\nu-1}\frac{1}{\mu}+\sum_{\mu=1}^{N-\nu}\frac{1}{\mu}-\sum_{\mu=\nu+1}^{N+\nu}\frac{1}{\mu}+\frac{1}{2\nu}\\
&= -\sum_{\mu=1}^{N+\nu}\frac{1}{\mu}+\frac{1}{\nu}+\sum_{\mu=1}^{N-\nu}\frac{1}{\mu}+\frac{1}{2\nu}\\
&= \frac{3}{2\nu}-\left\{\frac{1}{N-\nu+1}+\frac{1}{N-\nu+2}+\cdots+\frac{1}{N+\nu}\right\}.
\end{aligned}$$

When we substitute this into (5), we get

$$\begin{aligned}
&\sum_{\mu}\frac{1}{\mu^2}\sum_{\nu}\frac{1}{\nu^{2n-2}}+\cdots+\sum_{\mu}\frac{1}{\mu^{2n-1}}\sum_{\nu}\frac{1}{\nu^2} =\\
&(n+\tfrac{1}{2})\sum_{\nu}\frac{1}{\nu^{2n}}-\sum_{\nu}\frac{1}{\nu^{2n-2}}\left\{\frac{1}{N-\nu+1}+\frac{1}{N-\nu+2}+\cdots+\frac{1}{N+\nu}\right\}.
\end{aligned} \tag{7}$$

Finally,

$$0<\frac{1}{N-\nu+1}+\frac{1}{N-\nu+2}+\cdots+\frac{1}{N+\nu}<\frac{2\nu}{N-\nu+1},$$

and so

$$\begin{aligned}
0 &< \sum_{\nu}\frac{1}{\nu^{2n-1}}\left\{\frac{1}{N-\nu+1}+\cdots+\frac{1}{N+\nu}\right\}\\
&< 2\sum_{\nu}\frac{1}{\nu^{2n-2}}\cdot\frac{1}{N-\nu+1}\leqq 2\sum\frac{1}{\nu(N-\nu+1)} \qquad \left(n\geqq\frac{3}{2}\right)\\
&= \frac{2}{N+1}\sum\left\{\frac{1}{\nu}+\frac{1}{N-\nu+1}\right\}=\frac{4}{N+1}\sum\frac{1}{\nu}\\
&< \frac{4}{N+1}(1+\log N)\to 0 \qquad (N\to\infty).
\end{aligned} \tag{8}$$

Statements (7) and (8), taken together, complete the proof.

By successive applications of the convolution just established, one may now express $\zeta(2n)$ as a rational multiple of $\{\zeta(2)\}^n$. The theorem tells us nothing, however, about $\zeta(2)$, itself. We can circumvent this difficulty by considering an allied function,

$$\xi(s) \equiv \sum_{\nu=0}^{\infty} (-)^{\nu} \frac{1}{(2\nu+1)^s} \qquad (s > 0).$$

It, too, has a convolution, given by

THEOREM II:

$$\xi(1)\xi(2n-1) + \xi(3)\xi(2n-3) + \cdots + \xi(2n-1)\xi(1) = \left(n - \frac{1}{2}\right) \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^{2n}} \qquad (n = 1, 2, 3, \cdots).$$

Now, it is a well-known fact that

$$\xi(1) \equiv 1 - \frac{1}{3} + \frac{1}{5} - + \cdots = \arctan 1 = \frac{\pi}{4}.$$

Hence, taking $n=1$ in the above theorem, we find

$$\sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^2} = 2\{\xi(1)\}^2 = \frac{\pi^2}{8}.$$

Coupling this with the transparent identity,

$$\sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^s} = (1 - 2^{-s})\zeta(s) \qquad (s > 1),$$

the familiar value, $\zeta(2) = \pi^2/6$, emerges at once.

The proof of Theorem II proceeds along the same lines as that of Theorem I, though the details are a bit more intricate. In the first place, we have, as in (1)–(5) *supra*,

$$\sum_{\nu} \sum_{\mu} (-)^{\nu+\mu} \left\{ \frac{1}{2\mu+1} \cdot \frac{1}{(2\nu+1)^{2n-1}} + \cdots + \frac{1}{(2\mu+1)^{2n-1}} \cdot \frac{1}{2\nu+1} \right\}$$

(9)

$$= n \sum_{\nu} \frac{1}{(2\nu+1)^{2n}} + 2 \sum_{\nu} \frac{1}{(2\nu+1)^{2n-1}} \sum_{\mu}{}' (-)^{\nu+\mu} \frac{2\mu+1}{(2\mu+1)^2 - (2\nu+1)^2}.$$

Here, all sums run from 0 to $N$, and the accent has the same significance as before. Again,

$$4\sum_{\mu}{}' (-)^{\nu+\mu}\frac{2\mu+1}{(2\mu+1)^2-(2\nu+1)^2}$$

$$(10)\quad =\sum_{\mu}{}' (-)^{\nu+\mu}\frac{2\mu+1}{(\mu+\nu+1)(\mu-\nu)}=\sum_{\mu}{}' (-)^{\nu+\mu}\frac{1}{\mu-\nu}+\sum_{\mu}{}'(-)^{\nu+\mu}\frac{1}{\mu+\nu+1}$$

$$=-\frac{1}{2\nu+1}+(-)^{N-\nu}\left\{\frac{1}{N-\nu+1}-\frac{1}{N-\nu+2}+-\cdots+\frac{1}{N+\nu+1}\right\},$$

the last step involving reasoning analogous to (6). Equations (9) and (10) now yield

$$\sum_{\mu}(-)^{\mu}\frac{1}{2\mu+1}\sum_{\nu}(-)^{\nu}\frac{1}{(2\nu+1)^{2n-1}}+\cdots+\sum_{\mu}(-)^{\mu}\frac{1}{(2\mu+1)^{2n-1}}\sum_{\nu}(-)^{\nu}\frac{1}{2\nu+1}$$

$$(11)\quad =\left(n-\frac{1}{2}\right)\sum_{\nu}\frac{1}{(2\nu+1)^{2n}}$$

$$+\frac{1}{2}\sum_{\nu}(-)^{N-\nu}\frac{1}{(2\nu+1)^{2n-1}}\left\{\frac{1}{N-\nu+1}-\frac{1}{N-\nu+2}+-\cdots+\frac{1}{N+\nu+1}\right\}.$$

But,

$$0<\frac{1}{N-\nu+1}-\frac{1}{N-\nu+2}+-\cdots+\frac{1}{N+\nu+1}\leqq\frac{1}{N-\nu+1},$$

and so the absolute value of the last sum in (11) is

$$\leqq\sum_{\nu}\frac{1}{(2\nu+1)^{2n-1}}\cdot\frac{1}{N-\nu+1}\leqq\sum_{\nu}\frac{1}{(2\nu+1)(N-\nu+1)}\qquad(n\geqq 1)$$

$$(12)\quad =\frac{2}{2N+3}\sum_{\nu}\left\{\frac{1}{2\nu+1}+\frac{1}{2(N-\nu+1)}\right\}$$

$$=\frac{2}{2N+3}\left\{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2N+1}+\frac{1}{2N+2}\right\}\to 0\ (N\to\infty),$$

as with (8), above. From (11) and (12), the proof of Theorem II follows at once.

As we have already seen, the first convolution serves to give $\zeta(2n)$, recursively, as a rational multiple of $\{\zeta(2)\}^n$, which is $\{\pi^2/6\}^n$. Now, it may be shown (by, *e.g.*, one of the methods alluded to in our opening paragraph) that

$$(13a)\qquad \zeta(2n)=(-)^{n-1}\frac{2^{2n-1}B_{2n}\pi^{2n}}{(2n)!},$$

where these so-called Bernoulli numbers are the coefficients in the power-series expansion of

$$\frac{z}{e^z - 1} \equiv \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}. \tag{13b}$$

(*N.B.*, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$, and so on.) If there were any simple, closed expression for these numbers, (13) would represent an indisputable improvement over Theorem I; but the fact is that they themselves are most readily computed by means of recurrence relationships also. For example, (13b) may be stated symbolically as

$$\frac{z}{e^z - 1} = e^{Bz} \tag{13c}$$

whence,

$$ze^{-z} = e^{Bz} - e^{(B-1)z},$$

and, equating coefficients of like powers of $z$,

$$(B - 1)^n = B_n + (-)^n n. \tag{14}$$

Let us see now how our results fit in with this Bernoulli-number representation. Substituting from (13a), Theorem I becomes, after a little algebra,

$$\sum_{\nu=1}^{n-1} \binom{2n}{2\nu} B_{2\nu} B_{2n-2\nu} = -(2n + 1)B_{2n}. \tag{15}$$

From this point of view, then, the convolution is but a disguised recurrence equation for Bernoulli numbers. There is, of course, nothing to stop us from proving (15) directly from (13c), as follows. Differentiating,

$$\frac{1}{e^z - 1} - \frac{ze^z}{(e^z - 1)^2} = Be^{Bz},$$

$$\frac{z^2}{(e^z - 1)^2} = \frac{ze^{-z}}{e^z - 1} - Bze^{(B-1)z},$$

$$e^{(B+B)z} = e^{(B-1)z} - Bze^{(B-1)z},$$

$$\begin{aligned}(B + B)^n &= (B - 1)^n - Bn(B - 1)^{n-1} \\ &= (B - 1)^n - n(B - 1)(B - 1)^{n-1} - n(B - 1)^{n-1} \\ &= -(n - 1)(B - 1)^n - n(B - 1)^{n-1},\end{aligned}$$

and, by virtue of (14), we therefore have

$$-(B + B)^n = (n - 1)B_n + nB_{n-1}. \tag{16}$$

When $n$ is odd and greater than 1, this is trivial, since the right-hand side becomes simply $nB_{n-1}$ and on the left all the terms in the binomial expansion vanish except the first ones in, from both ends, *i.e.*,

$$- (nB_1B_{n-1} + nB_{n-1}B_1).$$

When $n$ is even, (16) is equivalent to (15). A similar reduction may be made of Theorem II by substituting for $\xi(2n+1)$; the latter is a rational multiple of $\pi^{2n+1}$, the coefficient involving an "Euler number."

In order to escape the imputation of producing nothing but new proofs of old facts, we conclude with a fresh result which flows from the method of this paper. By considering the convolution

$$\sum\sum\left\{\frac{1}{\mu^2}\cdot\frac{1}{\nu^{n-1}} + \frac{1}{\mu^3}\cdot\frac{1}{\nu^{n-2}} + \cdots + \frac{1}{\mu^{n-1}}\cdot\frac{1}{\nu^2}\right\},$$

one proves easily that

THEOREM III:

$$2\sum_{\nu=1}^{\infty}\frac{1}{\nu^n}\left(1+\frac{1}{2}+\cdots+\frac{1}{\nu-1}\right)$$
$$= n\zeta(n+1) - \{\zeta(2)\zeta(n-1) + \zeta(3)\zeta(n-2) + \cdots + \zeta(n-1)\zeta(2)\},$$
$$(n = 2, 3, 4, \cdots).$$

It was his efforts to find a closed form for the left side of the foregoing that led the author to the present considerations.

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## MATHEMATICAL NOTES

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### POWER SERIES AND ALGEBRAIC NUMBERS

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**Introduction.** The Lindemann theorem states that $ae^{\alpha}+be^{\beta}+\cdots+se^{\sigma}$ cannot vanish if $a, b, \cdots, s$ are algebraic numbers not all zero, and $\alpha, \beta, \cdots, \sigma$ are algebraic numbers no two of which are equal. In a recent paper [1] Dietrich and Rosenthal show how the Lindemann theorem can be used to prove the transcendence of certain power series:

THEOREM I (*Dietrich and Rosenthal*). *If $a_0, a_1, a_2, \cdots$ is a sequence of algebraic numbers, ultimately periodic but not ultimately all zero, then the value of the*

*function* $g(z)=\sum a_n z^n/n!$ *is transcendental when* $z$ *has an algebraic value other than zero.*

A similar use of the Lindemann theorem is made in [2].

**The theorem.** Now, the full force of the Lindemann theorem is needed to prove Theorem I, and yet it seems that the Lindemann theorem cannot be deduced from Theorem I. Thus, Theorem I and the Lindemann theorem are not equivalent. Our purpose here is to prove a more general result along the same lines which, as we shall show, is equivalent to the Lindemann theorem:

THEOREM II. *Suppose* $f(z)=\sum a_n z^n$ *is rational, is regular at* $z=0$, *has algebraic coefficients, and is not a polynomial. Let* $m$ *be the multiplicity of that root in the denominator of* $f$ *which has maximum multiplicity. Then (apart from a set of at most* $m$ *values of* $z$*) the value of the function* $g(z)=\sum a_n z^n/n!$ *is transcendental when the value of* $z$ *is algebraic.*

In Theorem II we have replaced the periodicity condition of I by a requirement that the $a_n$'s satisfy a linear recurrence relation. This rather minor alteration makes the result fit into the following more general context: Let $f(z)=\sum a_n z^n$, $h(z)=\sum b_n z^n$, and $g(z)=\sum a_n b_n z^n$, all with algebraic coefficients. If $f(z)$ is algebraic for algebraic values of $z$, and $h(z)$ is transcendental for algebraic $z\neq 0$, can anything be said about $g(z)$ for algebraic $z$? How must $f$ and $h$ be restricted to get a theorem in that direction? *Our result answers this question for* $f(z)$ *rational,* $h(z)=e^z$. The fact that this extremely special case is equivalent to the Lindemann theorem gives an idea of the depth of the general problem.

*Proof.* The proof of Theorem II has no mathematical depth, but it unites differential equations, difference equations, power series, and algebraic numbers in an entertaining manner.—To begin with, let $f(z)$ have denominator $z^p+bz^{p-1}+\cdots+rz+s$. Then multiplying and equating coefficients, as in [3], we see that $a_n+ba_{n+1}+\cdots+sa_{n+p}=0$ for all large $n$. Considering the derivatives of $g(z)$, as in [4], we find that $g(z)$ satisfies a differential equation $g^{(q)}+bg^{(q+1)}+\cdots+sg^{(q+p)}=0$ for some fixed $q$. Now, this is a linear differential equation with constant coefficients. Hence by the usual method of solution (which, one can prove, gives really *all* the solutions) we can write

$$g(z)=A(z)+B(z)e^{\beta z}+\cdots+S(z)e^{\sigma z} \tag{1}$$

where $\beta,\cdots,\sigma$ are the distinct roots of

$$1+bz+\cdots+sz^p=0 \tag{2}$$

and $A, B,\cdots, S$ are polynomials. The degree of $B,\cdots, S$ is one less than the multiplicity of the corresponding root.

As we see by the method given in [5], the algebraic character of the $a_n$'s ensures that of $b,\cdots, s$, and hence that of $\beta,\cdots,\sigma$. Since the polynomials $A, B,\cdots, S$ are found from the initial conditions $g^{(k)}(0)$, they must have alge-

braic coefficients, and since $g(z)$ is not a polynomial by hypothesis, at least one number $\beta, \cdots, \sigma$ and the corresponding polynomial $A, B, \cdots, S$ are not zero. The Lindemann theorem applied to the equation $g(z)=w$ for algebraic $z\neq 0$ and $w$ shows now that all of the coefficients $B, C, \cdots, S$ must vanish at this value of $z$. Since each of these polynomials has degree at most $m-1$, that situation can arise for at most $m-1$ points. Taking account of the case $z=0$, which is always exceptional, we obtain Theorem II.

**Discussion.** The fact that certain polynomial coefficients in (1) may be identically zero shows that *the m exceptional cases can actually occur.* If the coefficients ultimately form a periodic sequence, however, then the difference relation becomes simply $a_{n+p}=a_n$, so that (2) is $z^p=1$. Thus the exponents $\beta, \cdots, \sigma$ in (1) are the $p$th roots of unity, the polynomials $B, \cdots, S$ are constants, and $z=0$ is the only exceptional value. Alternatively, it is easy to compute $f(z)$ in this case, and the denominator is found to have no multiple roots; so $m=1$. Hence *Theorem II contains Theorem I.*

Again, the Lindemann theorem follows from Theorem II when we choose $f(z)=a/(1-\alpha z)+b/(1-\beta z)+\cdots+s/(1-\sigma z)$. Thus *Theorem II amounts to a rewording of the Lindemann theorem,* and an independent proof of the former would give the latter.

The theorem was suggested by [1], the proof by [3]–[5] and by conversation with Ernst Straus.

**References**

1. Dietrich and Rosenthal, Transcendence of Factorial Series with Algebraic Coefficients, Bull. Am. Math. Soc., v. 55, 1949, p. 55.
2. LeVeque, W. J., Note on the Transcendence of Certain Series, Proc. Am. Math. Soc., v. 2, 1951, No. 3, p. 401.
3. Pölya and Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. II, Dover Publications, N. Y., Ch. VII, Prob. 17.
4. *Ibid.*, Prob. 30.
5. *Ibid.*, Chap. VIII, Prob. 151.

## ON A CLASS OF IRRATIONAL NUMBERS

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The classical method of proving that $e$ is an irrational number is well-known. The following theorem generalizes the approach in the classical method.

THEOREM

1. *Let $a_n$, $n=1, 2, \cdots$ be positive or negative integers or zero, provided only that there be an infinite number of $a_n$ not equal to zero.*

2. *Let a number $S$ be defined by the series*

$$S=\sum_{n=1}^{\infty}\frac{a_n}{r^n(n!)^b}$$

*where r and b are any positive integers.*

3. *Suppose that a positive constant C and a positive constant $\alpha<b$ exist, independent of n, such that*

$$|a_n| < Cn^{\alpha}$$

*for all n greater than some number N. Then the number S is irrational.*

*Proof.* Suppose the number $S$ represented by the series is rational and equal to $p/q$ where $p$ and $q$ are taken to be relatively prime integers. Thus

$$\frac{p}{q} = \frac{a_1}{r(1!)^b} + \frac{a_2}{r^2(2!)^b} + \cdots + \frac{a_{Mq}}{r^{Mq}[(Mq)!]^b} + \cdots .$$

Here $M$ is an integer which we shall assume is larger than the number $N$ described in 3.

Multiplying both sides by $r^{Mq}[(Mq)!]^b$ we obtain

$$F = G + H,$$

where

$$F = \frac{p}{q} r^{Mq}[(Mq)!]^b$$

$$G = \left\{\frac{a_1}{r(1!)^b} + \cdots + \frac{a_{Mq}}{r^{Mq}[(Mq)!]^b}\right\} r^{Mq}[(Mq)!]^b$$

$$H = \frac{a_{Mq+1}}{r(Mq+1)^b} + \frac{a_{Mq+2}}{r^2(Mq+2)^b(Mq+1)^b} + \cdots .$$

Quantities $F$ and $G$ are integers. However

$$\begin{aligned} |H| &\leqq \frac{C}{r}\frac{(Mq+1)^{\alpha}}{(Mq+1)^b} + \frac{C}{r^2}\frac{(Mq+2)^{\alpha}}{(Mq+1)^b(Mq+2)^b} + \cdots \\ &< \frac{C}{r}\left[\frac{1}{(Mq+1)^{b-\alpha}} + \frac{1}{r(Mq+1)^{2b-\alpha}} + \cdots\right] \\ &\leqq \frac{C}{r}\frac{(Mq+1)^{\alpha}}{(Mq+1)^b - 1}, \end{aligned}$$

which can be made less than one by choosing $M$ large enough. Hence our supposition that the series represents a rational number is contradicted and so the theorem is proved.

*Remark.* The above theorem may be used to prove that sin 1, cos 1, $J_0(1)$ among many others, as well as $e$, are irrational numbers.

## A BRIEF PROOF OF A THEOREM ON $T$-TRANSFORMATIONS

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The transformation of the sequence $\{z_n\}$ by the matrix $(a_{n,k})$ into the sequence $\{z_n'\} \equiv \{\sum_{k=1}^{\infty} a_{n,k} z_k\}$ is called a $T$-transformation (and the matrix a $T$-matrix)* if it has the property that the transform of every convergent sequence also converges, and to the same limit. The well-known necessary and sufficient conditions for this are:†

$$\sum_{k=1}^{\infty} |a_{n,k}| < M \qquad \text{for } n > n_0 \text{ (fixed)}, \tag{1}$$

$$\lim_{n\to\infty} a_{n,k} = 0 \qquad \text{for each fixed } k, \tag{2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1. \tag{3}$$

The following theorem was first published by P. Dienes,‡ treating separately the case when $L$ and $U$ are (i) both finite, (ii) one infinite and (iii) both infinite. However, his treatment of the cases of infinite limit points was inadequate, and a proof of these cases was given by P. Vermes.§

The brief method given below treats all the cases simultaneously. The reader will notice its similarity to a construction of Agnew.‖

THEOREM. *Let $\{x_k\}$ be a real sequence with (finite or infinite) upper and lower limits $U$ and $L$, and let $L \leqq x \leqq U$. Then there is a non-negative $T$-transformation $(a_{i,j})$ such that $\lim_{n\to\infty} x_n' \equiv \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} x_k = x$ (finite or infinite).*

If $L = U$ (finite or infinite), $x = L = U$ is the only limit point of the sequence, so that the identical transformation will serve.

Otherwise, let $\{y_r\}$ be a sequence such that $\lim_{r\to\infty} y_r = x$ and $L < y_r < U$.

Then, since $L$ and $U$ are limit points of $\{x_k\}$, there exists a sequence $\{x_{k_p}\}$ of points of $\{x_k\}$ such that $x_{k_{2r-1}} < y_r < x_{k_{2r}}$ $(r = 1, 2, \cdots)$.

The matrix $(a_{i,j})$ given by

$$a_{i,j} = \begin{cases} \dfrac{x_{k_{2i}} - y_i}{x_{k_{2i}} - x_{k_{2i-1}}} & \text{when } j = k_{2i-1}, \\ \dfrac{y_i - x_{k_{2i-1}}}{x_{k_{2i}} - x_{k_{2i-1}}} & \text{when } j = k_{2i}, \\ 0 & \text{otherwise,} \end{cases}$$

* In America such a matrix is called "regular."

† Cooke, (1), 64.

‡ Dienes, (2), 391–2.

§ See Cooke, (1), 77–79.

‖ See Cooke, (1), 80.

is a non-negative $T$-matrix, and transforms the sequence $\{x_k\}$ into the sequence $\{y_r\}$ whose limit is $x$.

Thus any point in the closed interval $[L, U]$ is the generalized limit of the sequence by some non-negative $T$-matrix. On the other hand, no non-negative $T$-matrix can give a generalized limit outside this interval; in fact for every non-negative $T$-matrix, $L \leqq \underline{\lim}_{n\to\infty} x_n' \leqq \overline{\lim}_{n\to\infty} x_n' \leqq U$, by Knopp's core theorem.†

**References**

1. R. G. Cooke, *Infinite Matrices and Sequence Spaces* (Macmillan), 1950.
2. P. Dienes, *The Taylor Series* (Oxford), 1931.

† Cooke, (1), 138, (6.1, I).

## CURVES ENCIRCLING A CYLINDER

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Let $C$ be a curve encircling a right circular cylinder of unit radius in such a manner that each element of the cylinder contains a single point of the curve. We further require that $C$ have diameter 2, where by diameter we mean the least upper bound of the distance between points of $C$. It may not be immediately apparent that there exist curves other than the circular right cross-sections which have diameter equal to 2, but actually there is a considerable class of such curves, as appears shortly. It will also appear that $C$ is rectifiable, and we denote its length by $L$. By the height $H$ of $C$ we mean the minimum distance between two planes perpendicular to the cylinder and containing $C$ between them. We prove

$$2\pi \leqq L \leqq 2\sqrt{2}\,\pi, \tag{1}$$

$$0 \leqq H \leqq \sqrt{2}, \tag{2}$$

and the bounds given are the best possible ones.

In cylindrical coordinates, $C$ may be described by the equations $r=1$, $z=f(\theta)$. If $(1, \theta, f(\theta))$ and $(1, \phi, f(\phi))$ are points of $C$, we require that $[f(\theta)-f(\phi)]^2 + [2-2\cos(\theta-\phi)] \leqq 4$, or

$$|f(\theta) - f(\phi)| \leqq 2\left|\cos \tfrac{1}{2}(\theta - \phi)\right|. \tag{3}$$

We note that $f(\theta)-f(\theta+\pi)=0$; that is, $f$ has period $\pi$. Setting $\phi=\alpha+\pi$, we have from (3).

$$|f(\theta) - f(\alpha)| \leqq 2\left|\sin \tfrac{1}{2}(\theta - \alpha)\right|. \tag{4}$$

The equation (4) and the requirement that $f$ have period $\pi$ are sufficient as well as necessary conditions that $C$ have diameter 2, since the steps by which

(4) was obtained can be reversed. From (4) we have for every $\theta$, $\alpha$,

$$\left|\frac{f(\theta) - f(\alpha)}{\theta - \alpha}\right| \leqq \left|\frac{\sin \frac{1}{2}(\theta - \alpha)}{\frac{1}{2}(\theta - \alpha)}\right| < 1.$$

Thus $f$ is continuous, has its derivative quotients bounded by 1, and is in fact absolutely continuous, with $|f'| \leqq 1$ almost everywhere. Thus

$$2\pi = \int_0^{2\pi} d\theta \leqq L = \int_0^{2\pi} \sqrt{1 + (f'(\theta))^2}\, d\theta \leqq \int_0^{2\pi} \sqrt{2}\, d\theta = 2\sqrt{2}\,\pi,$$

and (1) is proved.

The lower bound of $L$ is obviously attainable. To show that the upper bound given is the best possible, we construct an example by piecing together segments of the function $2 \sin \frac{1}{2}\theta$ as follows. Divide the interval $(0, 2\pi)$ into $4n$ equal parts. In the first part, $0 \leqq \theta \leqq \pi/2n$, we define $f(\theta) = 2 \sin \frac{1}{2}\theta$. In the second part, $\pi/2n \leqq \theta \leqq \pi/n$, we define $f$ so as to make it an even function with respect to the point $\pi/2n$; that is, $f(\theta) = f((\pi/n) - \theta)$. In the remaining intervals, $f$ is defined so as to have period $\pi/n$. It is now not difficult to show that $f$ satisfies (4). In the first place, it is clear that we can restrict our attention to $\theta$ and $\alpha$ differing by less than $\pi$. By reducing angles modulo $\pi/n$ and noting that $\sin \frac{1}{2}\theta$ is an increasing function between 0 and $\pi$, we can reduce the problem to the cases where $\theta$ and $\alpha$ are in the same or adjacent intervals. In each of the several special cases that arise, the result follows readily from the increasing and concave nature of the function $\sin \frac{1}{2}\theta$.

In the first interval, $|f'| = \cos \frac{1}{2}\theta \geqq \cos(\pi/4n)$. By the periodicity and symmetry, $|f'| \geqq \cos(\pi/4n)$ in every interval, and

$$L = \int_0^{2\pi} \sqrt{1 + (f')^2}\, d\theta \geqq 2\pi\sqrt{1 + \cos^2(\pi/4n)}.$$

Thus $L$ can be made arbitrarily close to $2\sqrt{2}\pi$. An interesting question is whether or not there exists a $C$ whose length is exactly $2\sqrt{2}\pi$. If so, $f' = \pm 1$ almost everywhere and yet the derivative quotient is always less than 1, so that $f'$ is discontinuous everywhere.

The height $H$ is given by $\max f(\theta) - \min f(\theta)$. Assume that the minimum of $f$ comes at 0. Then by (3) and (4),

$$f(\theta) - f(0) \leqq 2 \min\left[\left|\sin \tfrac{1}{2}\theta\right|, \left|\cos \tfrac{1}{2}\theta\right|\right]. \tag{5}$$

The largest value of the right-hand member of (5) is $\sqrt{2}$, which occurs when $\theta = \pi/2$. However, this same right-hand member is a function $f$ satisfying (4) and having period $\pi$; in fact it is the same as the example constructed for length, with $n = 2$. Thus $\sqrt{2}$ is the best bound obtainable for $H$, and (2) is proved.

It would appear from the examples that it is impossible that simultaneously $H$ be close to $\sqrt{2}$ and $L$ close to $2\sqrt{2}\pi$. This suggests the interesting but perhaps difficult problem of finding the upper bound of $H + L$.

## CLASSROOM NOTES

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### INSIGHT AND UNDERSTANDING IN THE CALCULUS*

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Elementary calculus is, I think, one of the most difficult subjects to teach effectively. The concepts are unfamiliar to the student, hard to grasp and assimilate, and the techniques and applications often are involved and difficult. Even though he develops technical facility, often he feels he does not really understand the subject. His knowledge tends to be atomistic, consisting of individual topics and techniques loosely related to each other and individually unmotivated.

I think most teachers adopt one of two approaches to the situation. Some teachers, feeling the student can't really understand theory until he is more mature, concentrate on developing technique and problem-solving ability. Others, feeling that the student can't really understand the techniques of the subject without some knowledge of theory, and in order to avoid the need for unlearning in a later class in advanced calculus, put the emphasis on rigor of treatment.

I don't think either of these treatments is sufficient, in itself, to promote understanding of the calculus at the introductory level. The first tends to leave the student with particularistic knowledge of what to do in specific situations—but weak on general ideas and principles to guide him. As a result in a new or unfamiliar situation he clutches for a rule to apply rigidly. He achieves a feeling of security by setting up rules of thumb to be applied to the situations which ordinarily arise. Mathematics tends to degenerate into a mere tool for obtaining the correct answer to standard problems. The second approach, emphasizing rigor of treatment, tends to leave the student up in the air, since the abstractions studied are not rooted in concrete material. It is very doubtful that a student can appreciate a relatively rigorous treatment of the calculus if he does not first have a good understanding of the ideas at a more concrete and intuitive level.

I have a few points to make—they are not exhaustive—on how to foster understanding in elementary calculus. Most of them are not new—they are the sort of thing we piously agree upon as general principles but tend to disregard in the classroom under the pressure of achieving so-called practical results.

The first point I wish to make, and the most important, is that abstractions should be introduced on the basis of concrete experience. The student should see how the concepts arise from the attempt to cope with an aspect of experience, how they are suggested by concrete situations, so that he might almost feel he could have discovered them himself!

* Presented in a symposium on the teaching of calculus at the meeting of the Mathematical Association of America held at Brown University, December 29, 1951.

I think the most important application of this is to the introduction of the derivative. This is probably the most difficult notion the student encounters in the whole course—if he does not get some insight into this concept fairly early it will tend to color his reaction to the whole subject. I think it doubtful that a high order abstraction like derivative, which took some of the best minds of the human race centuries to develop, can be grasped on the basis of a brief introductory lecture on instantaneous rates which yields a four or five step method for calculating the result. If this is done, the derivative may seem to the student, not the culmination of years of thought on rates of change but the end result of a mysterious process of adding increments and dividing by $\Delta x$. In some texts we have, I think, an even worse situation—the derivative is defined without reference to instantaneous rates and the student discovers later, sometimes chapters later, that instantaneous rate is an "application" of the derivative. That is, the concrete materials which should motivate the abstraction appear later as applications of it. On the contrary, I think we should introduce the derivative as an abstract form of instantaneous rate on the basis of detailed study of the rate concept. I like to use the freely falling body as a central illustration, partly because the student thinks he understands the notion instantaneous speed in this case—he even knows the formula for it. I would have the student calculate average speeds in many specific intervals using the formula $s=16t^2$ so that he gets a good intuitive perception of the variation of average speed with the interval. Then I would bring up for discussion the instantaneous speed at $t=1$. We wrestle with the idea, analyze the "common sense" definitions suggested by the students, and show them inadequate. I would not give them the definition of instantaneous speed as a limit but try to elicit it from the physical situation. This can be done as follows. We do not know that there is such a thing as the speed of the falling body at $t=1$, but if there is such a speed $v$, it certainly must by physical intuition exceed the average speed in any interval terminating at $t=1$. Applying this to the interval $1-\Delta t$ to 1, we have by a simple calculation

$$32 - 16\Delta t < v.$$

Similarly physical intuition suggests $v$ must be less than the average speed in any interval starting at $t=1$. Thus we get

$$v < 32 + 16\Delta t.$$

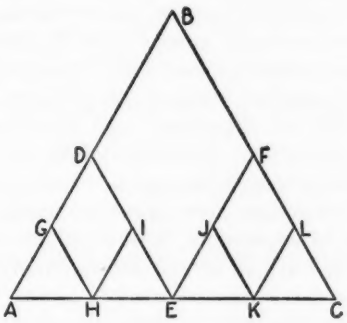
Considering these relations we see that $v$ must be greater than all speeds below 32 ft./sec. and must be less than all speeds above 32 ft./sec. Thus we are forced to single out 32 ft./sec. as the only possible value for the speed at $t=1$. This is very important since it exhibits the unique character of 32 ft./sec.—it is singled out as the common boundary or limiting value of all speeds in intervals preceding $t=1$ and of all speeds in intervals succeeding $t=1$, and it is not itself an average speed in any such interval. This discussion I think, not merely gives some substance to the notion instantaneous speed, but shows that the concept

of limit or boundary value is essential for its definition and provides a good motivation for studying the limit concept. It is evident that instantaneous speed can not be gotten by algebra—that is by finite calculation—and that a new concept involving infinite collections or infinite sequences is essential.

Furthermore, I think these same ideas should be examined in detail in geometrical terms—in a discussion of slope of an arc of a curve, slope of a curve at a point, tangent line, *etc.* These notions belong not merely to theoretical geometry, but have an honest practical importance for the student, since they are applicable to graphs of scientific data. Thus if we plot a distance-time graph for a moving body, slope of an arc represents average speed *etc.* This gives the student a chance to rethink the ideas in graphical form and better assimilate them.

I have been emphasizing the need for concrete examples in motivating a theory. Now I want to point out the importance of counterexamples. In the attempt to clarify ideas and motivate theory, counterexamples are of the utmost importance. I would be tempted to say that often it is more important to give counter-illustrations of an idea than illustrations of it. Often the student feels a very simple formulation of an idea or a theorem is adequate, and sees no reason for the complex technical formulation. Or he sees no reason for introducing an idea at all, because he can't seriously imagine the opposite. Apparently his intuitive reactions based as they are on limited experience do not encompass the situation in all its generality. Of course he doesn't realize this. Here an appropriate counterexample can do wonders. Thus, for example, students do not appreciate the importance of continuity because they have no real experience with discontinuous functions. The examples given in the texts tend to be rather academic—some are so remote from the student's experience that they carry little weight, like the function which is 0 for rational values and 1 for irrational values of the argument. Many examples do not really illustrate the point. To be specific, we wish to convince the student that he may fall into error, if he obtains the limit of a function at a point by merely taking the functional value at that point. The familiar examples of discontinuous functions $1/x$ and $\tan x$ do not illustrate this, since these functions are not defined at the alleged points of discontinuity—in fact they are continuous throughout their domain of definition. A good example to illustrate discontinuous behavior is familiar as a sort of puzzle. Consider equilateral triangle $ABC$ with each side of length 1. Join the midpoints of the legs to the midpoint of the base, forming two smaller triangles as in the diagram. Repeat the construction for the individual triangles formed and continue the process endlessly. Consider the sequence of paths: $ABC$, $ADEFC$, $AGHIEJKLC$, . . . and the corresponding sequence of their lengths. We ask for the limit of the sequence of lengths and the usual reply is that obviously the sequence of lengths approaches 1, the length of $AC$, as limit. Computing the lengths we see the first is 2, and the second equals the first since $DB=EF$ and $BF=DE$. Thus the second length is 2. Clearly all succeeding lengths equal 2 so that although the paths approach $BC$ as a limit, their lengths

do not approach 1, the length of $BC$, as a limit. This strongly challenges the student's cherished belief that variation is always continuous—he is shaken, disturbed, begins to feel his intuition is not infallible and that perhaps there is something to the method of reasoning after all!




Once a basic concept has been introduced, its significance should be reinforced by having the student operate with it as a part of his basic intellectual equipment—not with a formalized manipulative substitute which enables him to get correct results without consciously applying the idea. The example I am most concerned about is the calculation of the derivative of a function from its definition. I fear the familiar 4-step process for doing this tends to degenerate into rote procedure which the student applies mechanically, without consciously having the definition of derivative in mind. Thus to obtain the derivative of $f(x)=x^2$ he writes $y=x^2$, replaces $x$ by $x+\Delta x$, $y$ by $y+\Delta y$ to get $y+\Delta y=(x+\Delta x)^2$, subtracts the first equation from the second, divides by $\Delta x$, *etc.* I'm afraid there is not apt to be much present in the student's mind when he does this except that he must add an increment to whatever he sees in the first equation—I sometimes think there must be a guardian angel who inhibits the student from adding an increment to the exponent 2 or to the equal sign. I don't mean that a student can't consciously think of the significance of the steps as he applies this technique—rather I think there is no need to—the process is motivated essentially by mechanical considerations. It has further disadvantages. It tends to enforce the impression that the basic idea is derivative of a function, not derivative of a function at a particular point. I like to give students a problem like this: Find the derivative of $f(x)=(x+1)/(x-1)$ at $x=2$ directly from the definition. I find many students who use the 4-step process can't do this. They must first get the derivative for $x$, which is more difficult, then substitute $x=2$. Apparently in the 4-step process after substituting 2 for $x$ they are lost, since they do not have an $x$ to replace by $x+\Delta x$. Furthermore I do not think this process prepares a student properly to cope with problems which arise later on, like finding the derivative of a function defined geometrically as an area or arc

length, or that of a function like $x^2 \sin(1/x)$, defined as 0 at $x=0$, which is not given by a single analytic expression.

The final point I wish to make is that given two alternative formulations of a theory, we should choose the one which brings the ideas to the fore—it is not sufficient for a treatment merely to be scientifically correct or to have the merit of facilitating calculation. This is completely disregarded in the treatment of Integral Calculus given in many current texts. We have in Integral Calculus two important ideas: integration as summation, integration as anti-differentiation. The first is an outgrowth of mensuration problems which the student encounters in plane and solid school geometry and which has an honorable history of over 2,000 years in our subject. Mathematicians are still studying it, in abstract form, in the modern theory of measure. It is of the utmost importance in applied mathematics. Yet in the conventional treatment no name is assigned to summation to signalize its importance, least of all the name integration. It is introduced after weeks of work on formal anti-differentiation as an application of anti-differentiation, and is merely referred to as the "limit of a sum." Thus the fundamental theorem of Integral Calculus is expressed

$$\int_a^b f(x)dx = \lim \sum_i f(x_i)\Delta x_i$$

where the left member is the variation in an anti-derivative of $f(x)$ from $x=a$ to $x=b$. This tends to give students the impression that anti-differentiation is the essential idea in Integral Calculus, and that the limit of a sum is just an interpretation or application of this idea. It appears to the student that limits of sums have no life of their own except in relation to formal anti-differentiation, since he never solves a single problem of summation in its own terms. This whole treatment impedes the unification of the student's past ideas concerning mensuration and those he will get in advanced calculus and graduate school where mensuration ideas or integration in the strict sense come into their own.

Surely we should prefer the alternative treatment, in which the concept limit of a sum is defined and studied in its own terms, as an important abstraction from problems of measuring magnitudes arising in geometry and physical science, and is represented by the historical symbol of summation $\int_a^b f(x)dx$. To give the student experience with the idea he should evaluate some simple definite integrals directly from the definition. Then he can better appreciate the importance of the fundamental theorem as relating the two apparently disparate ideas summation and anti-differentiation. The fundamental theorem might well be expressed in the form

$$\int_a^b f(x)dx = \left[D^{-1}f(x)\right]_a^b$$

where $D^{-1}$ denotes the inverse of the differentiation operator. This then indicates the importance of anti-differentiation as a simple means of evaluating definite

integrals, and gives the student a motivation for weeks of study of formal antidifferentiation.

In conclusion I wish to emphasize that what I am advocating is not in the least opposed to raising standards of mathematical rigor or to developing technical facility. On the contrary, I think a good understanding of fundamental concepts and principles in concrete terms is a necessary core without which the student will not appreciate the need for raising standards of logical precision, and without which his technical work may degenerate into mere formalism.

## NON-ABELIAN GROUPS OF ORDER $pq$

S. K. BERBERIAN, Southern Illinois University

This note is concerned with the construction of non-abelian groups of order $pq$. The point of departure is the theorem characterizing such groups:*

THEOREM: *If $G$ is a non-abelian group of order $pq$, $p$ and $q$ primes, $p<q$, then:*

(1) $q\equiv 1 \pmod{p}$;
(2) *$G$ is generated by elements $P$ and $Q$, of orders $p$ and $q$, respectively, such that*
(3) $P^{-1}QP=Q^{\beta}$, where
(4) *$\beta$ is a primitive root of the congruence*
$x^p\equiv 1 \pmod{q}$.

It is evident that as soon as two elements $P$ and $Q$ satisfying (2), (3), and (4) are located in a known group, they generate a subgroup satisfying the requirements for $G$.

We therefore assume property (1) to hold, and attempt to locate suitable elements $P$ and $Q$ in a known group. For embedding purposes, we utilize the symmetric permutation group of degree $q$.

First, there are numbers $\beta$ satisfying the congruence (4); for, by assumption (1), the prime $p$ is a divisor of $q-1$, the order of the multiplicative group of the field of integers modulo $q$. Since such an element $\beta$ has order $p$, the full list of distinct solutions modulo $q$ of the congruence (4) is:

$$1, \beta, \beta^2, \cdots, \beta^{p-1}.$$

For $Q$, we choose the $q$-cycle

$$Q = (1, 2, \cdots, q).$$

Modulo $q$, we may write

$$Q^{\beta} = (1, 1+\beta, 1+2\beta, \cdots, 1+(q-1)\beta).$$

$P$ will necessarily be the product of disjoint $p$-cycles. In assumption (1), suppose that $q=1+mp$; we will choose $P$ to be the product of $m$ disjoint $p$-cycles. With such a structure, $P$ will have to leave one symbol of $Q$ unmoved; if we agree to let this symbol be 1 (this choice is arbitrary), then by (3), and the

* Burnside, W.: Theory of Groups of Finite Order; p. 48.

properties of conjugacy for permutations, $P$ must act on the remaining symbols of $Q$ according to the scheme:

$$\begin{array}{cc} Q = (1, & 2, \quad 3, \quad \cdots, \quad i, \quad \cdots, \quad q) \\ (5) & \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \end{array}$$

$$Q^\beta = (1, 1+\beta, 1+2\beta, \cdots, 1+(i-1)\beta, \cdots, 1+(q-1)\beta).$$

Thus, $P$ must send $i$ to $1+(i-1)\beta$, which in turn must be sent to $1+[1+(i-1)\beta-1]\beta=1+(i-1)\beta^2$. A typical cycle of $P$ would then be

$$(6) \qquad \pi = (1+(i-1), 1+(i-1)\beta, 1+(i-1)\beta^2, \cdots, 1+(i-1)\beta^{p-1}).$$

The cycle $\pi$ terminates after $p$ entries, since $\beta^p \equiv 1 \pmod q$, and hence

$$1+(i-1)\beta^{p-1} \to 1+(i-1)\beta^p \equiv i \pmod q.$$

The application of this principle is very direct.

*Example* 1: $p=3$, $q=7$

A primitive solution of the congruence $x^3 \equiv 1 \pmod 7$ is $\beta=2$. Choose

$$Q = (1\ 2\ 3\ 4\ 5\ 6\ 7).$$

Then,

$$Q^\beta = (1\ 3\ 5\ 7\ 2\ 4\ 6).$$

Assuming again that $P$ leaves the symbol 1 fixed, we simply force $P$ to meet the requirements:

$$P = (2\ 3\ 5)(4\ 7\ 6).$$

Then,

$$P^{-1}QP = Q^2$$

and $P$ and $Q$ generate the required group $G$.

We next show that the construction of a $P$, satisfying (3), is always possible. Example 1 suggests a method of proof. For the first cycle of $P$, write

$$\pi_1 = (1+1, 1+\beta, 1+\beta^2, \cdots, 1+\beta^{p-1}).$$

Since $1, \beta, \beta^2, \cdots, \beta^{p-1}$ are incongruent modulo $q$, so are the entries of $\pi_1$. The cycling is proper, since the symbol $1+\beta^{p-1}$ of $Q$ must be sent to

$$1+\beta^p \equiv 1+1 \pmod q.$$

At this point, if $q=3$, then $p=2$ and the construction is ended. Otherwise, there is a symbol $i \neq 1$ of $Q$ which does not appear in $\pi_1$; for the next factor of $P$, construct

$$\pi_2 = (1+(i-1), 1+(i-1)\beta, 1+(i-1)\beta^2, \cdots, 1+(i-1)\beta^{p-1}).$$

It has been already noted in connection with (6) that the cycling of $\pi_2$ is proper.

The entries of $\pi_2$ are distinct modulo $q$, for, an equality

$$1 + (i - 1)\beta^r \equiv 1 + (i - 1)\beta^s \pmod{q}$$

would imply

$$\beta^{r-s} \equiv 1 \pmod{q},$$

and since $0 \leqq r, s < p$, and $\beta$ has order $p$, it follows that $r = s$. Also, $\pi_2$ is disjoint from $\pi_1$; for suppose there were an equality

$$1 + (i - 1)\beta^r \equiv 1 + \beta^s \pmod{q}.$$

This would imply

$$i \equiv 1 + \beta^{s-r} \pmod{q}$$

contrary to the choice of $i$.

If there is a third step in the construction,

$$\pi_3 = (1 + (j - 1), 1 + (j - 1)\beta, \cdots, 1 + (j - 1)\beta^{p-1})$$

where $j \neq 1$ is neither in $\pi_1$ nor $\pi_2$, the preceding argument shows that $\pi_3$ is disjoint from $\pi_1$. To show that $\pi_3$ is disjoint from $\pi_2$, suppose there were an equality

$$1 + (j - 1)\beta^r \equiv 1 + (i - 1)\beta^s \pmod{q}.$$

Then,

$$j \equiv 1 + (i - 1)\beta^{s-r} \pmod{q}$$

whereas $j$ is not in $\pi_2$.

Finally, $P = \pi_1\pi_2 \cdots \pi_m$, the construction terminating in exactly $m$ steps, since there are $q-1$ symbols of $Q$ to be moved, and $q - 1 = mp$. (An explicit induction step has been omitted; the argument surrounding $\pi_3$ is nearly adequate.) We have done little more than construct the system of prime residues modulo $q$.*

*Example* 2: The case $p = 2$ is especially transparent, since we are then forced to choose $\beta = q - 1$. If then $Q = (1, 2, \cdots, q)$, $Q^\beta$ will be the inverse of $Q$:

$$Q^\beta = (1, q, q - 1, \cdots, 2),$$

and if 1 is the symbol unmoved by $P$, then

$$P = (2, q)(3, q - 1)(4, q - 2) \cdots .$$

For example, if $q = 7$, then

$$Q = (1\ 2\ 3\ 4\ 5\ 6\ 7)$$
$$Q^\beta = Q^6 = (1\ 7\ 6\ 5\ 4\ 3\ 2)$$
$$P = (27)(36)(45).$$

* Uspensky, J. V., and Heaslet, M. A.: Elementary Number Theory; p. 225.

This representation of the non-abelian group of order $2q$ naturally suggests the dihedral group. Let the vertices of a regular $q$-gon be numbered in cyclic order. In the dihedral group for this polygon, $Q$ may be interpreted as a rotation of $360°/q$, and $P$ as a reflection in the axis of symmetry passing through vertex 1. Since $q$ is odd, every reflection symmetry leaves fixed exactly one vertex; our choice of 1, as the symbol unmoved by $P$, may therefore be interpreted as a choice of vertex.

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## ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

*Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.*

### PROBLEMS FOR SOLUTIONS

E 1046. *Proposed by E. P. Starke, Rutgers University*

Knowing that $dy/dx$ and $dx/dy$ are reciprocals, students sometimes jump to the conclusion that $d^2x/dy^2$ should be the reciprocal of $d^2y/dx^2$. Determine functions $y=f(x)$ for which this is indeed true.

E 1047. *Proposed by W. R. Ransom, Tufts College*

Find all Gaussian integral factorizations of $10i$, where $i=\sqrt{-1}$.

E 1048. *Proposed by H. S. Shapiro, Massachusetts Institute of Technology*

Let $M$ be an $n\times\infty$ matrix. Show that if every $n\times n$ submatrix of $M$ is singular, then the columns of $M$ are linearly dependent.

E 1049. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

If a uniform thin rod of length $L$ is perpendicular to and touching the earth's surface at one end, find the distance between the center of gravity and the center of mass of the rod.

E 1050. *Proposed by S. H. Gould, Purdue University*

If $c_0=1$, $c_{i+1}=2^{c_i}$, prove that $\sum_{i=1}^{\infty} c_i^{-1}$ is transcendental.

## SOLUTIONS

### Area-and-Perimeter Bisectors of a Quadrilateral

E 992 [1951, 699; 1952, 408]. *Proposed by Kaidy Tan, Anglo-Chinese College, Amoy, China*

Draw a straight line which will bisect both the area and the perimeter of a given (convex) quadrilateral.

*Further discussion by W. B. Carver, Cornell University.* Michael Goldberg's solution of this problem [1952, 408] raises at least as many questions as it answers, and his statement that "there are an odd number of area-and-perimeter bisectors unless there are an infinity of them" is not quite correct.

To carry through Goldberg's method of solution one would set up his equations (1) and (2) for a pair of opposite sides and obtain from them two pairs of values of $x$ and $y$ as tentative *possible* solutions of the problem. (When $x$ and $y$ are unequal, interchanging them gives a different pair and a possible *different* solution of the problem.) When equations (1) and (2) are set up for the other pair of opposite sides, two more tentative solutions are found. Similarly his equations (4) and (5) must be set up for two pairs of adjacent sides, yielding four more tentative solutions. From these eight tentative solutions one must then pick out the ones, certainly less than eight, which give the actual solutions of the geometric problem. As Goldberg shows, there must always be at least one solution of the problem, and *all* solutions of the problem (when the number is finite) must be found among the eight tentative solutions obtained from his equations.

To look into the matter further we consider first the case of a quadrilateral of which no two sides are parallel. For any such quadrilateral the vertices may be lettered $A$, $B$, $C$, $D$ in counter-clockwise order and in such a way that $BA$ and $CD$ meet at $Q$, $BC$ and $AD$ meet at $R$, with the points on the lines in the orders $BAQ$, $CDQ$, $BCR$, $ADR$. We may denote the lengths of the sides by $a=AB$, $b=BC$, $c=CD$, $d=DA$; and we let $m=AQ$, $n=DQ$, $t=CR$. Then we have

$$a < b + c + d, \qquad b < c + d + a, \tag{1}$$

$$a + b > c + d. \tag{2}$$

If $b+c<a+d$, we may turn the quadrilateral over, interchanging $A$ and $C$; hence there is no loss of generality in assuming that

$$b + c \geqq a + d. \tag{3}$$

We denote the area of the quadrilateral by $2K$ and the perimeter by $2s$.

Two points $P$ and $P'$ on the perimeter may be called *opposite* points when they bisect the perimeter; and we let $A'$, $B'$, $C'$, $D'$ be the points opposite to the respective vertices $A$, $B$, $C$, $D$. Because of the inequalities (1), (2), (3), these eight critical points on the perimeter will always occur in the counter-clockwise

order $AC'BD'A'CB'D$, with $B$ and $D'$, and therefore $B'$ and $D$, coinciding in the special case when $b+c=a+d$.

Now let a variable point $P$, starting at $A$, move around the perimeter in the counter-clockwise direction, the distance from $A$ to $P$, measured around the perimeter, being denoted by $x$. The variable area inside the quadrilateral on one side of the line $PP'$ will be a continuous function of $x$, say $\phi(x)$, where $\phi(0)=$area of triangle $ABA'$ and $\phi(s)=$area of quadrilateral $A'CDA$. If we set $f(x)=\phi(x)-K$, we have an area-and-perimeter bisector whenever $f(x)=0$ in the interval $0\leqq x<s$. We then have

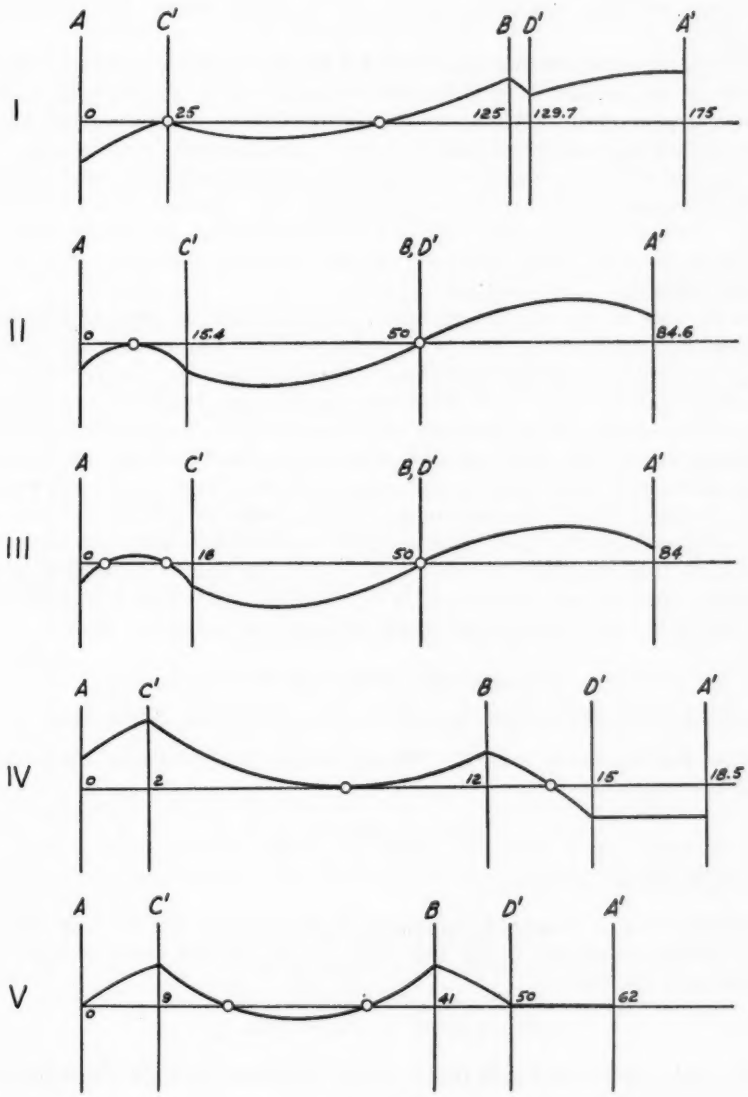
| For $P$ in the interval | $x$ | $f(x)=\phi(x)-K$ |
|---|---|---|
| $AC'$ | $0\leqq x\leqq a+b-s$ | $\frac{1}{2}(\sin B)(a-x)(s-a+x)-K$ |
| $C'B$ | $a+b-s\leqq x\leqq a$ | $\frac{1}{2}(\sin Q)[(m+a)(m+c)+mn-2(m+x)(m+s-d-x)]$ |
| $BD'$ | $a\leqq x\leqq s-d$ | $\frac{1}{2}(\sin C)(a+b-x)(s-a-b+x)-K$ |
| $D'A'$ | $s-d\leqq x\leqq s$ | $\frac{1}{2}(\sin R)[2(t+a+b-x)(r+d-s+x)-(r+d)(t+b)-rt]$ |

Thus for each of the four intervals between the critical points the function $f(x)$ is of the form $Ex^2+Fx+G$, with $E<0$ in the first, third, and fourth intervals, and $E>0$ in the second interval. Hence the graph of $y=f(x)$ consists of four arcs of parabolas, convex downward in the second interval and convex upward in the other three intervals. The function $f(x)$ is continuous throughout the interval $0\leqq x\leqq s$, but the derivative $f'(x)$ is discontinuous at the critical points. Also we have $f(0)=-f(s)$.

Goldberg's statement that the number of area-and-perimeter bisectors is odd when it is finite overlooks the possibility that the graph of $y=f(x)$ may just *touch* the $x$-axis at a critical point or at a maximum or minimum point of one of the parabolic arcs. Such a point of tangency corresponds to a "double" root in a purely algebraic sense, but it indicates *only one* area-and-perimeter bisector. The accompanying graphs correspond to certain definite numerical cases. A convex quadrilateral is uniquely determined by giving the lengths of its sides, (in order) and the cosine of one angle.

Graph I is for the quadrilateral $a=125$, $b=75$, $c=75+4\sqrt{55}$, $d=75-4\sqrt{55}$, $\cos B=4/5$. For this quadrilateral $s=175$, $K=3000$. There are just two area-and-perimeter bisectors; the line $CC'$ is one, and the other occurs for a value of $x$ approximately 86.

Graph II is for the symmetric quadrilateral $a=b=50$, $c=d=25\sqrt{29}-100$, $\cos B=7/25$. We have $s=25\sqrt{29}-50$, $K=75(33-4\sqrt{29})$. Again there are just two area-and-perimeter bisectors; the axis of symmetry $BD$ is one, and a line $PP'$ perpendicular to the axis of symmetry with $AP=x=(150-25\sqrt{29})/2$ is the other. The parabolic arc for the interval $AC'$ touches the $x$-axis at the midpoint of this interval.


I
A
C'
B
D'
A'
0
25
125
129.7
175
II
A
C'
B, D'
A'
0
15.4
50
84.6
III
A
C'
B, D'
A'
0
18
50
84
IV
A
C'
B
D'
A'
0
2
12
15
18.5
V
A
C'
B
D'
A'
0
9
41
50
62

GRAPHS I–V

For the symmetric quadrilateral $a=b=50$, $c=d=34$, $\cos B=7/25$ (Graph III), there are three solutions; the axis $BD$ and two others, $PP'$, for $AP=x=8\pm\sqrt{14}$.

For the case of the trapezoid we take $b=BC$ as the longer, and $d=DA$ as the shorter, of the parallel sides. Then the inequalities (1), (2), (3) hold, and all of the above treatment holds except that for the interval $D'A'$ the graph will be a straight line segment parallel to the $x$-axis. If the trapezoid is not isosceles this line segment will not coincide with the $x$-axis, and all solutions must occur in the three intervals from $A$ to $D'$. Graph IV is for the trapezoid $a=12$, $b=130\sqrt{6}-310$, $c=13$, $d=130\sqrt{6}-315$, with $A$ and $B$ right angles; $s=130\sqrt{6}-300$ and $K=15(52\sqrt{6}-125)$. Again there are just two solutions, one for $x=13\sqrt{6}-24$, and the other for $x$ approximately 13.7.

In the case of the isosceles trapezoid, $DAD'A'$ is a rectangle and we have solutions $PP'$ for $P$ at any point of the segment $D'A'$; but we may ask whether there are any other solutions in addition to these obvious ones. Graph V is for the isosceles trapezoid $a=c=41$, $b=30$, $d=12$, $\cos B=9/41$. In addition to the solutions for the interval $D'A'$ there are two other solutions for $x=(50\pm\sqrt{122})/2$.

Except for the cases of the parallelogram and the isosceles trapezoid the number of solutions is finite. The special cases show that there can be one, two, or three solutions. What is the maximum possible number of solutions? The maximum number is hardly four, because, given a case of four solutions, one could apparently alter it slightly to give a case of five. For reasons depending on the convexity upwards and downwards of the four parabolic arcs, I believe there could never be more than five solutions. Is the maximum five or three?

### A Formula Yielding Palindromic Numbers

E 1013 [1952, 249]. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Show that for $a=0, 1, 2, \cdots, 10$ and for any positive integral $n$ except 1 and 3, the expression

$$(10^n-1)(909a+1)$$

yields palindromic numbers.

*Solution by R. Z. Vause, University of North Carolina.* For $a=0$ ,the expression is simply a sequence of 9's. For $a=1, 2, \cdots, 10$, the factor $909a+1$ can be written in the form

$$r(10^3)+s(10^2)+r(10)+(s+1),$$

where $r$ and $s$ are integers such that $r+s=9$. Therefore, for $n\geqq 4$, the expression $(10^n-1)(909a+1)$ becomes

$$r(10^{n+3})+s(10^{n+2})+r(10^{n+1})+s(10^n)+9(10^{n-1})+\cdots$$
$$+9(10^4)+s(10^3)+r(10^2)+s(10)+r,$$

whereas for $n=2$ it is

$$r(10^5) + s(10^4) + s(10) + r.$$

Also solved by A. R. Hyde, H. Kaufman and J. S. Shipman (jointly), and the proposer.

Kaufman and Shipman pointed out two directions for generalization. In the first place the 909 may be replaced by any number of $2k+1$ digits of the form $9090 \cdots 909$. It can then be shown that

$$(10^n - 1)(9090 \cdots 909a + 1)$$

yields palindromic numbers for $a=0, 1, \cdots, 10$ and $n>2k+1$, and also for $n=2, 4, \cdots, 2k$. In the second place the problem, and the above generalization, can be extended in the obvious way to an arbitrary base $B$.

The proposer has neatly generalized his formula as follows. Let $a_i$, $i=1, 2, \cdots, n$, be $n$ arbitrary digits, with $a_1 \neq 0$, and define $b_i = 9-a_i$. Then, if we define $R_n$ by the sequence of digits

$$R_n = a_1 a_2 \cdots a_n b_n b_{n-1} \cdots b_1,$$

the number

$$P_n = (10^m - 1)(R_n + 1)$$

is a palindrome of $m+4n$ digits for all $m \geqq 2n$.

### An Impossible Arithmetic Triangle

E 1014 [1952, 249]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle having sides $a$, $b$, $c$ and area $S$, if the ratio

$$(a^2 + b^2 + c^2)/4S$$

is a whole number $m>1$, then the sides $a$, $b$, $c$ cannot all be integral.

*Solution by C. V. Fronabarger, Southwest Missouri State College.* Suppose $a$, $b$, $c$ are integral. Since $m$ is a constant for similar triangles, only three cases need be considered:

Case I, the three sides all odd;
Case II, only one side odd;
Case III, only one side even.

Now if

$$\begin{aligned}(a^2 + b^2 + c^2)/4S &= (a^2 + b^2 + c^2)/\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \\ &= m, \text{ an integer,}\end{aligned}$$

two conditions must be satisfied. First,

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 16S^2$$

must be a perfect square, and second, 4 must be a divisor of $a^2+b^2+c^2$.

In Cases I and II, $16S^2 \equiv 3 \pmod 4$, which is impossible, since all perfect squares are congruent to 0 or 1 modulo 4.

In Case III, $a^2+b^2+c^2 \equiv 2 \pmod 4$, and is thus not divisible by 4.

It follows that $a$, $b$, $c$ cannot all be integers.

Also solved by Louisa Grinstein, Vern Hoggatt, R. R. Phelps, and C. S. Venkataraman.

*Editorial Note*: There is nothing in the above proof which says that we cannot have $m=1$. Actually, however,

$$(a^2+b^2+c^2)/4S = \cot \omega,$$

where $\omega$ is the Brocard angle of the triangle. Since $\omega \leqq 30°$, $\cot \omega \geqq \sqrt{3}$, and it follows that $m \neq 1$. For the special case $m=2$ see problem 4401 [1952, 112].

To show, in Cases I and II, that

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) \equiv 3 \pmod 4$$

we observe, setting $d=a+b+c$, that

$$\begin{aligned}(a+b+c)&(-a+b+c)(a-b+c)(a+b-c)\\ &= d(d-2a)(d-2b)(d-2c) \equiv d^4 - d^3(2a+2b+2c) \equiv -d^4 \equiv 3 \pmod 4\end{aligned}$$

Cases I and II can be eliminated, as was done by Phelps, by using the fact that in a primitive Heronian triangle just one side must be even (see Wm. Fitch Cheney, Jr., *Heronian Triangles*, this MONTHLY [1929, 22–28]).

### An Integral Involving a Bessel Function

E 1015 [1952, 249]. *Proposed by C. A. Shook and Albert Wilansky, Lehigh University*

Evaluate

$$\int \frac{dx}{x[J_n(x)]^2},$$

where $J_n$ is the usual Bessel function.

*Solution by W. J. Klimczak, Trinity College.* Let $Y_n$ denote the Bessel function of the second kind. Then, for all values of $n$,

$$\frac{d}{dx}\left[\frac{Y_n(x)}{J_n(x)}\right] = \frac{J_n(x)Y_n'(x) - Y_n(x)J_n'(x)}{[J_n(x)]^2} = \frac{2}{\pi x[J_n(x)]^2}.$$

Hence

$$\int \frac{dx}{x[J_n(x)]^2} = \frac{\pi}{2}\frac{Y_n(x)}{J_n(x)} + C.$$

This formula is derived on p. 133 of G. N. Watson's *A Treatise on the Theory of Bessel Functions*. It also occurs as prob. 9, p. 184, of Ince's *Ordinary Differential Equations*.

Also solved by A. E. Danese, Vern Hoggatt, H. Kaufman and J. S. Shipman (jointly), M. S. Klamkin, David Mandelbaum, O. E. Stanaitis, and the proposers.

*Editorial Note*: If $n$ is not an integer it can be shown that

$$\int \frac{dx}{x[J_n(x)]^2} = -\frac{\pi}{2\sin n\pi}\frac{J_{-n}(x)}{J_n(x)}.$$

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## ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

*Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.*

### PROBLEMS FOR SOLUTION

4518. *Proposed by Paul Erdös, National Bureau of Standards, and Mark Kac, Cornell University*

For each integer $n$ let $\sigma_k(n)$ be the sum of the $k$th powers of its divisors. It is a natural conjecture that

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!}$$

is irrational. Prove this for the case $k=2$. (The case $k=1$ is problem 4493 [1952, 412].)

4519. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$1 + \frac{1-n}{1+n}\cdot\frac{1}{3} + \frac{(1-n)(2-n)}{(1+n)(2+n)}\cdot\frac{1}{5} + \cdots = \frac{1}{4n}\cdot\left\{\frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n-1)}\right\}^2.$$

4520. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $\phi(x)$ be a Lebesgue measurable function in the interval $0 \leqq x \leqq 1$ such that

$$\int_0^1 \phi(x)dx = 0, \qquad \int_0^1 x\phi(x)dx = 1.$$

Show that $|\phi(x)| \geqq 4$ in some set of measure greater than zero.

4521. *Proposed by I. A. Barnett, University of Cincinnati*

Find the complete solution of the diophantine equation

$$x^4 + y^4 + 64 = z^4.$$

4522. *Proposed by O. E. Stanaitis, St. Olaf College, Northfield, Minn.*

Evaluate

$$\lim_{k\to\infty} \int_{2k\pi}^{(2k+1)\pi} \phi(t^\alpha) \sin t dt,$$

where $\phi(x) = x - [x]$, $\alpha > 1$, and $[x]$ denotes the greatest integer in $x$.

## SOLUTIONS

### A Minimum Property of the Eigenvalues of a Hermitian Transformation

4429 [1951, 194]. *Proposed by Ky Fan, University of Notre Dame*

Let $H$ be a non-negative (positive semi-definite) Hermitian transformation in the $n$-dimensional unitary space. If the eigenvalues $\lambda_i (1 \leqq i \leqq n)$ of $H$ are arranged in ascending order: $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$, show that, for any positive integer $k \leqq n$, the product $\lambda_1\lambda_2 \cdots \lambda_k$ of the first $k$ eigenvalues is equal to the minimum of the expression

$$\prod_{i=1}^{k} (Hx_i, x_i),$$

when $k$ orthonormal vectors $x_1, x_2, \cdots, x_k$ vary in space. Here $(Hx_i, x_i)$ denotes the inner product of the vector $Hx_i$ with $x_i$. (This result is related to the Proposer's paper, *On a theorem of Weyl concerning eigenvalues of linear transformations, I*, Proceedings of the National Academy of Sciences, U.S.A., v. 35 (1949), pp. 652–655.)

*Solution by Hans Schneider, Edinburgh, Scotland.* Let the unitary space be spanned by the $n$ orthonormal vectors $x_1, x_2, \cdots, x_n$. Let $X$ be the unitary matrix having these $n$ vectors as columns, and let $X^*$ be its conjugate transposed matrix. Then the matrix $X^*HX$ has the same eigenvalues as $H$. Now let $H_k$ denote the matrix obtained by deleting the last $n-k$ rows and columns of $X^*HX$ for $k = 1, 2, \cdots, n$. It is clear that for $k = 1, 2, \cdots, n$, $H_k$ is also a non-

negative definite Hermitian matrix.

Denote the eigenvalues of $H_k$ by $\lambda_1^{(k)}, \cdots, \lambda_k^{(k)}$. It is known that the eigenvalues of $H_{k+1}$ are separated by those of $H_k$ and thus it follows that

$$\lambda_1 \cdots \lambda_k \leqq \lambda_1^{(k)} \cdots \lambda_k^{(k)} = \det (H_k).$$

But the determinant of a non-negative definite Hermitian matrix is less than the product of its diagonal elements. The diagonal elements of $H_k$ are precisely the numbers $(Hx_i, x_i)$, $i=1, 2, \cdots, k$, and thus we have

$$\prod_{i=1}^{k} \lambda_1 \cdots \lambda_k \leqq \prod_{i=1}^{k} (Hx_i, x_i). \tag{1}$$

If the $x_i$ are chosen to be the eigenvalues of $H$, the equality holds in (1) and the proposed result therefore follows.

Also solved by Richard Arens, Bernard Vinograde, Burton Wendroff, K. G. Wolfson, and the Proposer.

*Note by the Proposer.* Using the theorem stated in problem 4429, one can easily prove the following theorem: *Let $A$ be a linear transformation in the $n$-dimensional unitary space. Let the eigenvalues $\{\lambda_i\}$, $\{\rho_i\}$ of $A$ and $(A+A^*)/2$ respectively be so arranged that*

$$R\lambda_1 \leqq R\lambda_2 \leqq \cdots \leqq R\lambda_n, \qquad \rho_1 \leqq \rho_2 \leqq \cdots \leqq \rho_n.$$

*If the Hermitian transformation $(A+A^*)/2$ is non-negative (i.e. if all $\rho_i \geqq 0$), then for any positive integer $k \leqq n$:*

$$\prod_{i=1}^{k} \rho_i \leqq \prod_{i=1}^{k} R\lambda_i. \tag{2}$$

This result is a completion of a recent theorem by Ostrowski-Taussky (A. M. Ostrowski and O. Taussky, *On the variation of the determinant of a positive definite matrix*, Proc. Kon. Nederl. Akad. v. Wetensch. Amsterdam, Series A, v. 54 (1951), pp. 383–385). Under the same hypothesis as in the above theorem, these authors proved the inequality

$$\det \frac{A + A^*}{2} \leqq |\det A|, \tag{3}$$

which may be written $\rho_1\rho_2 \cdots \rho_n \leqq |\lambda_1\lambda_2 \cdots \lambda_n|$ and is therefore a consequence of the case $k=n$ of (2).

*Proof.* By Schur's superdiagonal form of matrices, there exist $n$ orthonormal vectors $x_1, x_2, \cdots, x_n$ such that $(Ax_i, x_i) = \lambda_i$, $1 \leqq i \leqq n$. Then we have

$$\left(\frac{A + A^*}{2} x_i, x_i\right) = \frac{\lambda_i + \overline{\lambda_i}}{2} = R\lambda_i, \qquad (1 \leqq i \leqq n),$$

and

$$\prod_{i=1}^{k}\left(\frac{A+A^*}{2}x_i, x_i\right) = \prod_{i=1}^{k} R\lambda_i, \qquad (1 \leqq k \leqq n).$$

Since $(A+A^*)/2$ is non-negative, according to the theorem of problem 4429, the left side is $\geqq \rho_1\rho_2 \cdots \rho_k$. Thus (2) is proved.

**Eigenvalues of a Sum of Hermitian Matrices**

4430 [1951, 194]. *Proposed by Ky Fan, University of Notre Dame*

Let $H'$, $H''$ be two $n$-rowed non-negative Hermitian matrices, and let $H=\alpha H'+\beta H''$, where $\alpha$, $\beta$ are two non-negative numbers with $\alpha+\beta=1$. Let the eigenvalues of $H'$, $H''$ and $H$ be denoted by $\lambda'_i$, $\lambda''_i$ and $\lambda_i (1 \leqq i \leqq n)$ respectively and so arranged that

$$\lambda'_i \leqq \lambda'_{i+1}, \qquad \lambda''_i \leqq \lambda''_{i+1}, \qquad \lambda_i \leqq \lambda_{i+1}.$$

Prove that for any positive integer $k \leqq n$:

$$\lambda_1\lambda_2 \cdots \lambda_k \geqq (\lambda'_1\lambda'_2 \cdots \lambda'_k)^\alpha(\lambda''_1\lambda''_2 \cdots \lambda''_k)^\beta, \tag{1}$$

and in particular:

$$\det H \geqq (\det H')^\alpha(\det H'')^\beta. \tag{2}$$

*Solution by the Proposer.* Let $\phi_i (1 \leqq i \leqq n)$ be an orthonormal set of eigenvectors of $H$: $H\phi_i=\lambda_i\phi_i$. Then

$$\lambda_1\lambda_2 \cdots \lambda_k = \prod_{i=1}^{k}(H\phi_i, \phi_i) = \prod_{i=1}^{k}[\alpha(H'\phi_i, \phi_i) + \beta(H''\phi_i, \phi_i)]$$

and therefore

$$\lambda_1\lambda_2 \cdots \lambda_k \geqq \prod_{i=1}^{k}(H'\phi_i, \phi_i)^\alpha(H''\phi_i, \phi_i)^\beta.$$

Now by a minimum property of the eigenvalues of non-negative Hermitian transformations (see problem 4429 above), the right side of the last inequality is not less than

$$(\lambda'_1\lambda'_2 \cdots \lambda'_k)^\alpha(\lambda''_1\lambda''_2 \cdots \lambda''_k)^\beta.$$

(1) is thus proved. For the particular case (2), another proof can be found in K. Fan, *On a theorem of Weyl concerning eigenvalues of linear transformations, II*, Proc. Nat. Acad. Sci. U.S.A., 36 (1950), pp. 31–34.

Also solved by Hans Schneider, and Bernard Vinograde.

**Words Having No Part Repeated $n$ Times**

4454 [1951, 569]. *Proposed by R. C. Lyndon, Princeton University*

The "Burnside problem" for semigroups without cancellation can be given the following form. Let $S(n, m)$ be the set of all words, or finite sequences, formed from an alphabet of $m$ letters, which have the property that no (non-empty) part is repeated as many as $n$ consecutive times. Show that, for $m, n > 1$, with the single exception $m = n = 2$, the set $S(n, m)$ is infinite.

*Comment by William Gustin, Indiana University.* The set $S(2, 2)$ in two letters $a, b$ obviously consists of the following six words: $a$, $ab$, $aba$, $b$, $ba$, $bab$; so is finite. That all other sets $S(n, m)$ are infinite has been shown by Morse and Hedlund (*Unending chess, symbolic dynamics, and a problem in semi-groups*, Duke Mathematical Journal, v. 11 (1944), pp. 1–7).

They exhibit an infinite word $B$ in two letters $a, b$ by defining its initial word intervals $B_k$ of length $2^k$ inductively as follows. Let $B_0 = a$; having defined $B_k$, define $B_{k+1} = B_k\overline{B}_k$ where $\overline{B}_k$ is obtained from $B_k$ by interchanging the letters $a$ and $b$. Thus

$$B = abbabaabbaababba \cdots .$$

It is shown that $B$ contains no word interval of the form $pQpQp$ where $p$ is a letter and $Q$ is a (possibly null) word. In particular $B$ is cube free, that is, contains no non-null word interval of the form $EEE$. Therefore each of the infinitely many initial word intervals of $B$ belongs to $S(n, m)$ for $n \geqq 3$, $m \geqq 2$.

The infinite word $B$ in two letters $a, b$ is transformed into an infinite word $C$ in three letters $a, b, c$ by destroying all letter squares $aa$, $bb$ in $B$ in the following way. Each letter in $B$ which is not followed by the same letter is left unchanged; each letter in $B$ which is followed by the same letter is changed to $c$. Thus

$$C = acbabcacbcabacb \cdots .$$

It is shown that $C$ is square free, that is, contains no non-null word interval of the form $EE$. Therefore each of the infinitely many initial word intervals of $C$ belongs to $S(n, m)$ for $n \geqq 2$, $m \geqq 3$.

Also solved by R. P. Dilworth, W. R. Scott, F. B. Thompson, and the Proposer.

*Editorial Note.* The Proposer wishes to correct the impression that the systems $S(n, m)$ are semigroups. In fact, the present problem contributes little to the solution of the Burnside problem for semigroups. (Burnside's conjecture is that every group on a finite number, $m$, of generators, in which every element except the unit element is of the same finite order, $n$, is necessarily a finite group. This conjecture has been confirmed only for a handful of cases.)

**Tetrahedron—Sum of Analogous Ratios**

4456 [1951, 570]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If through the vertices $A, B, C, D$ of a tetrahedron parallel planes are drawn cutting a given line $L$ in points $A_2, B_2, C_2, D_2$, and if $A_1, B_1, C_1, D_1$ are the points in which the lines $AA_2, BB_2, CC_2, DD_2$ cut the planes $BCD, CDA, DAB, ABC$, then

$$\frac{AA_2}{AA_1}+\frac{BB_2}{BB_1}+\frac{CC_2}{CC_1}+\frac{DD_2}{DD_1}=2.$$

*Solution by W. E. Byrne, Virginia Military Institute.* Taking a system of cartesian coordinates with oblique axes we may put $D(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$. Let line $L$ be given by

$$x = x_0 + \lambda l, \qquad y = y_0 + \lambda m, \qquad z = z_0 + \lambda n,$$

and let plane $\Pi_D$ through $D$ be $ux+vy+wz=0$. Then $D_2$ is the point on $L$ for which the parameter $\lambda$ satisfies

$$ux_0 + vy_0 + wz_0 + \lambda(lu + mv + nw) = 0.$$

Since plane $ABC$ has equation $x/a+y/b+z/c=1$, we have

$$\frac{1}{\mu}\left[\frac{x_0}{a}+\frac{y_0}{b}+\frac{z_0}{c}-\left(\frac{l}{a}+\frac{m}{b}+\frac{n}{c}\right)\left(\frac{ux_0 + vy_0 + wz_0}{lu + mv + nw}\right)\right] = 1$$

where $\mu=DD_2/DD_1$. Thus

$$\frac{DD_2}{DD_1}=\frac{x_0}{a}+\frac{y_0}{b}+\frac{z_0}{c}-\left(\frac{l}{a}+\frac{m}{b}+\frac{n}{c}\right)\left(\frac{ux_0 + vy_0 + wz_0}{lu + mv + nw}\right). \tag{1}$$

Plane $\Pi_A$ is $u(x-a)+vy+wz=0$ and plane $BCD$ is $x=0$. Hence for $A_2$

$$u(x_0 - a) + vy_0 + wz_0 + \lambda(lu + mv + nw) = 0$$

so that

$$\frac{AA_2}{AA_1}=\frac{-(x_0-a)}{a}+\left(\frac{l}{a}\right)\frac{u(x_0+a)+vy_0+wz_0}{lu+mv+nw}, \tag{2}$$

and similarly

$$\frac{BB_2}{BB_1}=\frac{-(y_0-b)}{b}+\left(\frac{m}{b}\right)\frac{ux_0+v(y_0-b)+wz_0}{lu+mv+nw}, \tag{3}$$

$$\frac{CC_2}{CC_1}=\frac{-(z_0-c)}{c}+\left(\frac{n}{c}\right)\frac{ux_0+vy_0+w(z_0-c)}{lu+mv+nw}. \tag{4}$$

Addition of equations (1)–(4) gives at once

$$\sum \frac{AA_2}{AA_1} = 3 - \left(\frac{lu + mv + nw}{lu + mv + nw}\right) = 2.$$

Also solved by Josef Langr.

**Coefficients in a Series of Cosines**

4457 [1951, 570]. *Proposed by Israel Halperin, Queens College, Kingston, Ontario*

If

$$\sum_{m=1}^{\infty} |b_m| < \infty \quad \text{and} \quad \lim_{t\to 0} \sum_{m=1}^{\infty} b_m \cos\frac{1}{mt} = 0,$$

show that every $b_m = 0$.

*Solution by the Proposer.* By actual integration

$$\int_{r\pi}^{(r+1)\pi} \left| \sum_{m=1}^{N} b_m \cos n_m u \right|^2 du = \frac{1}{2}\pi \sum_{m=1}^{N} |b_m|^2,$$

where $N$, $r$ are any positive integers and the $n_m$ are different integers. Hence

$$\text{l.u.b.}_{r\pi\leq u\leq (r+1)\pi} \left| \sum_{m=1}^{N} b_m \cos n_m u \right| \geqq \sqrt{\frac{1}{2}\sum_{m=1}^{N} |b_m|^2}.$$

Choosing $n_m = N!/m$ and setting $t = (N!)u$ gives

$$\text{l.u.b.}_{r(N!)\pi\leq t\leq (r+1)(N!)\pi} \left| \sum_{m=1}^{N} b_m \cos\frac{t}{m} \right| \geqq \sqrt{\frac{1}{2}\sum_{m=1}^{N} |b_m|^2}.$$

Since $\sum |b_m|^2$ is convergent and $\sum b_m \cos(t/m)$ is uniformly convergent, this implies that

$$\overline{\lim_{t\to\infty}} \left| \sum_{m=1}^{\infty} b_m \cos\frac{t}{m} \right| \geqq \sqrt{\frac{1}{2}\sum_{m=1}^{\infty} |b_m|^2}$$

and hence $\sum_{m=1}^{\infty} b_m \cos(1/mt) \to 0$ as $t \to 0$ only if all $b_m$ are zero.

**An Expansion Related to the Lagrange Interpolation Formula**

4458 [1951, 636]. *Proposed by Z. A. Melzak, McGill University, Montreal*

Let $f(x)$ be a polynomial of degree $n$. Then

$$f(x + y) = \frac{y(y+1)(y+2)\cdots(y+n)}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{f(x-k)}{y+k}.$$

I. *Solution by V. D. Gokhale, Atlanta University.* Consider the fraction

$$\frac{f(x+y)}{y(y+1)(y+2)\cdots(y+n)}.$$

Since the degree of the denominator is at least one higher than that of the numerator, this fraction decomposes into partial fractions of the form

$$\frac{A_k}{y+k},$$

$$A_k = \left[\frac{f(x+y)}{g'(y)}\right]_{y=-k}, \qquad g(y) = y(y+1)\cdots(y+n).$$

Thus

$$A_k = \frac{f(x-k)}{(-k)(-k+1)\cdots(-1)\cdot 1\cdot 2\cdots(n-k)} = \frac{(-1)^k f(x-k)}{k!(n-k)!}$$
$$= \frac{(-1)^k}{n!}\binom{n}{k} f(x-k),$$

so that

$$\frac{f(x+y)}{y(y+1)\cdots(y+n)} = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(x-k)}{y+k}.$$

Multiplying both sides by $y(y+1)\cdots(y+n)$ we get the desired result.

II. *Solution by W. V. Parker, Alabama Polytechnic Institute.* Each member is a polynomial in $y$ of degree $n$. It is readily seen that they are equal for $y=-k$, $k=0, 1, 2, \cdots, n$. Hence they are identical polynomials in $y$.

The right member can be obtained immediately from Lagrange's interpolation formula. In fact, for $f(x)$ a polynomial of degree $n$, $g(y) = \Pi_{i=0}^{n}(y-y_i)$, where the $y_i$ are distinct, we have

$$f(x+y) = g(y)\sum_{k=0}^{n}\frac{f(x+y_k)}{(y-y_k)g'(y_k)}.$$

Also solved by H. L. Alder, T. M. Apostol, Gerald Berman, Leonard Carlitz, W. C. Foreman, Emil Grossman, H. W. Gould, M. S. Klamkin, Walter Littman, A. E. Livingston, George Millman, F. D. Parker, M. Perisastri, H. N. Shapiro, O. E. Stanaitis, Chih-yi Wang, and the Proposer.

**Squarefull Integers**

4459 [1951, 636]. *Proposed by D. J. Newman, Harvard University.*

Find an asymptotic expression for the number of integers, not exceeding $x$, each of which has the property that each of its prime divisors divides it to the second power at least.

*Solution by Abe Sklar, University of Chicago.* We call such integers squarefull and denote the number of squarefull integers not exceeding $x$ by $A(x)$. If $n$ is squarefull and $m^2$ is the largest square divisor of $n$, then $d=n/m^2$ divides $m$ and $d$ is squarefree. Conversely, any squarefull integer can be written as a product of a square $m^2$ and a squarefree divisor $d$ of $m$. Hence:

$$A(x) = \sum_{m \leq x^{1/2}} \sum_{d} |\mu(d)|,$$

where the inner sum extends over all divisors of $m$ for which the product $m^2d$ does not exceed $x$; and $\mu(d)$ is the Möbius function, whose absolute value is 1 for squarefree arguments, zero otherwise. Changing the order of summation, we have:

$$A(x) = \sum_{d \leq x^{1/3}} \sum_{m} |\mu(d)|,$$

where the inner sum now extends over all multiples of $d$ for which the product $m^2d$ does not exceed $x$. There are $[(x/d^3)^{1/2}]$ such multiples. Hence:

$$\begin{aligned} A(x) &= \sum_{d \leq x^{1/3}} |\mu(d)| \, [(x/d^3)^{1/2}] \\ &= x^{1/2} \sum_{d \leq x^{1/3}} |\mu(d)| \, d^{-3/2} + O(x^{1/3}) \\ &= x^{1/2} \sum_{d=1}^{\infty} |\mu(d)| \, d^{-3/2}\{1 + O(x^{-1/6})\} + O(x^{1/3}) \\ &= x^{1/2} \sum_{d=1}^{\infty} |\mu(d)| \, d^{-3/2} + O(x^{1/3}). \end{aligned}$$

The coefficient of $x^{1/2}$ may be written in several forms, since

$$|\mu(d)| = \prod_{p} (1 + p^{-3/2}) = \frac{\zeta(3/2)}{\zeta(3)}.$$

Also solved by Robert Breusch, Leonard Carlitz, H. N. Shapiro, Abe Sklar (a second solution), and the Proposer.

## RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

*All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.*

*Advanced Calculus.* By Wilfred Kaplan. Cambridge, Mass., Addison-Wesley Press, Inc., 1952. 13+679 pages. $8.50.

This book which covers the subject matter generally included in a course in advanced calculus contains sufficient material for a course of two, three, or even four semesters. After a brief review of algebra, analytic geometry and calculus, the author devotes chapters 1, 3 and 5 to a notable treatment of vectors, and chapters 2 and 4 to the differential and the integral calculus respectively of functions of several variables. Chapter 6 contains a particularly fine treatment of series, chapter 7 is concerned with Fourier series and orthogonal functions, chapters 8 and 10 with ordinary and with partial differential equations respectively, while chapter 9 contains a long (131 pages) and excellent treatment of complex variable theory. Scattered through the book are many problems, ranging from the simple and straightforward to those which will tax the best students.

The author has an excellent style: definitions are clearly stated, new concepts are discussed and well motivated, important results are generally presented in the form of theorems, proofs are lucid and contain a sufficient amount of detail, theorems are stated formally in italics before their proofs are given, and the hypotheses of a theorem are never buried in the body of the proof.

Aside from a few typographical errors (pp. 23, 221, 276, 429) and a misleading drawing (fig. 1–16), the reviewer found only three statements in the book to which he can object, and these are trivial matters of phraseology. They are the statements on p. 56: "the vector $du/dt$ represents the tangent to the curve . . . ", on p. 195: "$R_1$ and $R_2$ overlapping only at boundary points," and on p. 307: "In general, the lower limit cannot exceed the upper limit". The dearth of errors of all kinds and the excellence of the format redound to the credit of both author and publisher and should greatly enhance the popularity of the book.

For the student planning to specialize in mathematics the book has a serious deficiency. He receives no training in the $\epsilon$, $\delta$ technique, learns nothing about the real number system or the fundamentals of the Riemann integral, and does not encounter the Heine-Borel theorem or the Bolzano-Weierstrass theorem. The statement in the preface that "A competent teacher can easily fill in these gaps, if so desired, and thereby present a complete course in real analysis" should not go unchallenged. A typical engineering student, on the other hand,

who is seeking some mathematics beyond the calculus will probably regard the book as too theoretical in spite of its applications, its numerical methods, and its extensive treatment of vectors.

L. J. GREEN
Case Institute of Technology

*Advanced Statistical Methods in Biometric Research.* C. R. Rao. New York. John Wiley & Sons, Inc. 1952. xvii+390 pp. $7.50.

Many books dealing with statistics emphasize either statistical theory (for those who are interested in the mathematical and logical bases of the subject) or statistical methods (for those who are interested primarily in the applications of statistics to some particular field). This cleavage of the subject is partly due to the training and interests of the authors, partly due to the backgrounds and interests of the two groups of readers, and partly due to the fact that a thorough development of the theory, an adequate explanation of its application, and a suitable demonstration of appropriate computations are apt to result in a work which is bulky and which lacks the desired unity of style.

The purpose of the author, as indicated in the preface and as revealed throughout the volume, was "to present a number of statistical techniques, keeping in view the requirements of both the student who questions the basis of a particular method employed and the practical worker who seeks a recipe for the reduction of data." He has hence provided whatever theoretical material (usually of a mathematical nature) was necessary for a precise statement and mathematical solution of each problem, and, in many cases, detailed computations illustrating the various steps in the numerical solution.

The book, as its title suggests, is primarily in the "methods" class rather than in the "theory" class, though the theory is by no means neglected. The first 85 pages are devoted to necessary mathematical and statistical material with little reference to "methods." The rest of the book is in a general way devoted to "methods" though additional suitable theoretical material (including several appendices) is included as needed. The "methods" part of the book features historical background, with lists of appropriate references and generous discussions of the nature of the problems under consideration, as well as the recommended solution.

The book is written primarily for workers in biometric research but will be of value to workers in other fields who are interested in the general area of multivariate analysis and the particular area of discriminatory analysis.

The material of the last two chapters dealing with the use of multiple measurements in problems of biological classification is of especial interest. An objective method which minimizes the errors of classification is introduced in the first of these chapters. The development leads to a problem which is similar to such linear programming problems as the Hitchcock-Koopmans transportation problem and the problem of differential prediction. The final chapter features discriminatory topology, which is useful in ordering statistical groups, and in

general the interrelationships between a number of populations or groups of individuals. Especially featured is the concept of group distance, introduced by Mahalanobis.

Each chapter has its own list of references. There are no problems in each chapter, except for the author's illustrations, but a list of 20 miscellaneous problems, most of which are of a theoretical nature, is given as an appendix.

The mathematician may be especially interested in the first 85 pages where the basic multivariate material through the derivation of the Wishart distribution is presented in concise form.

This book seems to the reviewer to be a valuable addition to the literature of statistical methods and statistical theory. The author has achieved considerable success in unifying the study of mathematical distributions, statistical inference, and computational methods and in presenting the result in a single volume.

P. S. Dwyer
University of Michigan

*Elementary Differential Equations*. By E. D. Rainville. The Macmillan Company, 1952. xii+392 pp. $5.00.

The text "A Short Course in Differential Equations" by the same author and publisher has been extended and supplemented to form the new "Elementary Differential Equations." This was done by using all of the plates from the Short Course, with no revision, to form the first thirteen chapters of the new text. The author, no doubt, saw a need for some revision and supplementing in this material and at least partially accomplished this by the appendage of three supplements. This procedure was a desirable economy from the publisher's point of view.

The introduction of elementary concepts in differential equations to the average, or lower-ability, student is accomplished in a remarkable manner. Necessary theory to give this student a proper concept is very clearly illustrated, avoiding the many fine points that would add to his confusion. Plenty of exercises are available to assure proficiency in the mechanics of solving differential equations. The inclusion of the Miscellaneous Exercises, all word problems, at the end of Supplement A is certainly a welcome addition.

The new chapters, fourteen through twenty-one, are written with the same care and viewpoint. An average student should be able to continue through this text and end with as many "right ideas" as could be expected for an introductory course.

Teachers will, in any course, want to introduce their own peculiar "likes" in the presentation of theory or new subject matter, and some are at times inclined to mention their "dislikes." I find the latter urge less frequent in this text. The answer to practically every problem is given with the problem. Some would prefer that the answers be listed at the end of the book; others that only

half of the answers be given. We all agree, I think, that where the answers are given, they be correct. To find a wrong answer in this text should almost be considered an accomplishment.

K. H. STAHL
University of Colorado

*Econometrics.* By Gerhard Tintner. New York, John Wiley and Sons. 1952. 370 pages. $3.75.

Econometrics, as the empirical counterpart of economic theory, bears about the same relation to mathematical economics as does experimental physics to mathematical physics. But controlled experiment is replaced by the collection and analysis of economic data, and statistical techniques rather than apparatus are the tools of observation. Whereas mathematical economics is over a century old, econometrics is a development of the last few decades and is now mushrooming with the encouragement of business, government, and the military. It is, inevitably, in a somewhat chaotic state and in need of summary and synthesis. Professor Tintner's book is a welcome contribution to answering this need and a very useful survey of the development and current status of econometrics.

Without attempting to be exhaustive, Prof. Tintner carefully discusses a considerable number of important methods, illustrates them with examples chosen from the literature or worked out by himself from real data, and gives extensive bibliographic references. The first part, which occupies only about one-fifth of the book, gives an introduction that presupposes only some acquaintance with economic theory, and, except for a short sketch of regression methods in Chapter 2, only a minimal mathematical background. The remainder of the book, devoted to multivariate analysis and time series, requires background in modern economics, calculus, and mathematical statistics. There is an appendix on elementary matrix theory and related numerical methods.

One of the best features of the book is that the author frankly recognizes and explicitly mentions the difficulties that arise from lack of data and of mathematical tools to handle even such data as are available. He points to the deviations of assumptions from actual conditions, indicates the resulting approximations and drawbacks, and suggests the required cautions. Particularly in the field of time series analysis there are numerous interesting indications of work to be done. This mature approach coupled with the worked out examples and references makes the book unusually and commendably alive and closely related to current significant work.

Although the writing is generally quite clear, there are a few places where terms are defined only by context and others where the reader is left to puzzle out important distinctions. For example, "identification" and "overidentification" are not explicitly defined in their technical senses in Chapter 7, and the purposes of discriminant analysis have to be distinguished and inferred from examples in Chapter 6. There are unsatisfactory statements about the solution

of linear homogeneous equations on pages 104, 116, 127, and 338—the first and last resulting from misplacing "only." The definition of a scalar as "a quantity which is not a matrix" on page 333 is unfortunate. Calculations to as many as nine significant digits (p. 211) seem inappropriate where both data and method are crude. Readers with strong views on the foundations of probability may be irritated by the author's ambiguous position based on a leaning toward the ideas of Keynes and Jeffreys coupled with a recognition that the frequency approach is at present the only usable tool. The complete lack of diagrams is surprising since there are many places where they would help the reader. These, however, are minor weaknesses and do not alter this reviewer's opinion that the book is a fine job, the best available introductory survey of econometrics, and an essential reference for anyone working in the field.

K. O. May
Carleton College

## NEW BOOKS RECEIVED

*Sir Isaac Newton, Opticks.* Foreword by Albert Einstein. Introduction by Sir Edmund Whittaker, Preface by I. B. Cohen. New York, Dover Publications. $1.90.

*Algebraic Projective Geometry.* By J. G. Semple and G. T. Kneebone. New York, Oxford University Press, 1952. $7.00.

*Linear Algebra and Matrix Theory.* International Series in Pure and Applied Mathematics. By R. R. Stoll. New York, McGraw-Hill Book Co., 1952. $6.00.

*Practical Calculus,* Revised Second Edition. By C. I. Palmer and C. E. Stout. New York, McGraw-Hill Book Company, 1952. $6.00.

*A Guide to Tables of the Normal Probability Integral* (Applied Mathematics Series 21). By National Bureau of Standards. August 1952, 16 pages, 15 cents.

*Differential Equations.* By Robert C. Yates. New York, McGraw-Hill Book Co., 1952. vii+215 pages. $3.75.

*Philosophy of Natural and Mathematical Sciences.* By Sister M. Helen Sullivan. New York, Vantage Press, Inc., 230 W. 41 Street. 1952. $3.75.

*Mental Prodigies.* By Fred Barlow. New York, Philosophical Library, 1952. 256 pages. $4.75.

*Description of a Magnetic Drum Calculator* (Annals, vol. 25). By the Computation Laboratory of Harvard University, 1952. 318 pages. $8.00.

*General College Mathematics,* First Edition. By W. L. Ayres, C. G. Fry and H. F. S. Jonah. New York, McGraw-Hill Book Co., 1952. xii+283 pages. $3.75.

*Methods of Applied Mathematics.* By F. B. Hildebrand. New York, Prentice Hall, 1952. xi+523 pages. $7.75.

*Fundamental Procedures of Financial Mathematics.* By Merrill and Irene Rassweiler. New York, The MacMillan Company, 1952. 8+260 pages. $3.25.

## CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

*Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.*

### SUMMER WORK FOR MATHEMATICS STUDENTS

The Board of U. S. Civil Service Examiners for Scientific and Technical Personnel of the Potomac River Naval Command announced on October 28, 1952, an examination for Student Aid Trainee in Chemistry, Metallurgy, Physics, Mathematics, or Engineering. This examination will be used to recruit college students for both a cooperative education plan and a summer employment plan. In the summer employment plan, a student is employed only during the summer vacation period, and attends college during the entire regular college term. For further information, see Announcement No. 4-34-4 (1952) available at any first or second-class post office, except in cities where a U. S. Civil Service Regional Office is located.

### CLUB NEWS

(Clubs are invited to contribute newsworthy items.)

Kappa Mu Epsilon, national honorary mathematics society, reports the following chapter activities:

Alabama Beta, State Teachers College, is making plans to be co-hosts with the College for a meeting of the Alabama Teachers of College Mathematics.

Indiana Alpha, Manchester College, prepared an exhibit of mathematical equipment and models.

Kansas Alpha at State Teachers College, Pittsburg, and Missouri Alpha at Southwest Missouri State College, Springfield, have arranged an exchange of programs. On April 17, four student papers were given at Springfield by members of Kansas Alpha. This Fall a similar program will be given at Pittsburg by members of Missouri Alpha.

Michigan Gamma, Wayne University, announces the following Mathematics Awards on the basis of a competitive examination: Richard Pauley, first place; Joseph Garrity, second place; Duane Morrow, third place; Robert Reibel and David Morrison, honorable mention. William Shulevitz and Max Krolik were awarded student memberships in the Mathematical Association of America.

Miss Zelia Zulauf of Missouri Beta, Central Missouri State College, Warrensburg, presented a paper, *Mathematics and music*, at the St. Louis meeting of the Missouri Academy of Science.

Morris Rosen was awarded the prize given by New York Alpha, Hofstra College, to the student ranking highest in the first year of mathematics.

Ruth Rickloff of Pennsylvania Alpha, Westminster College, was awarded the K.M.E. prize. This prize is presented annually to a second-year student in mathematics on the basis of his entire academic record.

Tennessee Alpha, Polytechnic Institute, Cookeville, awarded the Mathematics Medal to Ira F. Grissom at the 1952 graduation exercises.

### O. U. MATH LETTER

The University of Oklahoma Chapter of Pi Mu Epsilon is sponsoring the *O. U. Mathematics Letter*, a four-page mimeographed publication designed particularly for high-school students. It is sent free about twice a semester to mathematics teachers who request it. The issue of September, 1952, contains brief discussions of "Angle trisection" and "What mathematics to take in high school," as well as a "Problem Box" and other items of interest.

Many other clubs sponsor similar publications. This department would be pleased to receive sample copies so that a check list of the various club periodicals can be made.

### LIGHT—BUT NOT TRIVIAL—PROBLEM

The following problem proposed by Professor Norman Anning may be of interest: Find where these two curves intersect.

$$y = x(x-1)(x-2)(x-3), \qquad y = (x^2 - 3x + 1)^2.$$

"The student who cheerfully eliminates $y$ will at the same time eliminate $x$ and get a surprise. The curves do not intersect at any finite point. At infinity on $x=0$, both curves have the same sort of super-singularity which $y^3=x^4$ has at the origin."

### ON SOLVING CUBIC EQUATIONS

NORMAN ANNING, University of Michigan

We teach two methods for solving cubic equations. It is the purpose of this note to smooth the transition from one method to the other. What Bombelli in 1572 called *casus irreducibilis* remains irreducible in the sense that a cubic which has three real roots cannot be solved by algebra alone.

It is suggested that, after a cubic has been prepared for solution, it be identified with

$$x^3 - 3m^2x - 2n^3 = 0.$$

The roots of this are $p+q$, $p\omega+q\omega^2$, $p\omega^2+q\omega$, where

$$\omega^2 + \omega + 1 = 0 \tag{1}$$

$$p^3 = n^3 + \sqrt{n^6 - m^6} \tag{2}$$

$$q = m^2/p. \tag{3}$$

There are three real roots if $n^6 < m^6$. In this "case" let us put $n^3 = m^3 \cos 3A$. The equation becomes the trisection equation and the student by a simple application of DeMoivre's theorem can transform the roots given by "Cardan" into

$x_1 = 2m \cos A, \quad x_2 = 2m \cos (A + 120°), \quad x_3 = 2m \cos (A + 240°).$

For instance,

$$x = \sqrt[3]{n^3 + \sqrt{n^6 - m^6}} + \sqrt[3]{n^3 - \sqrt{n^6 - m^6}}$$

becomes

$$x/m = \sqrt[3]{\cos 3A + i \sin 3A} + \sqrt[3]{\cos 3A - i \sin 3A} = 2 \cos A.$$

Please note that, if we put $x = ny$ in our chosen form of cubic, we shall get the equation treated on pages 20–30 of the addenda to the Jahnke-Emde *Tables*.

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## NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.*

### MICHIGAN MATHEMATICAL JOURNAL

A new medium for publication of mathematical research, the *Michigan Mathematical Journal*, has been initiated. The editorial board, formed of members of the faculty of the University of Michigan, is as follows: R. Brauer, W. Kaplan, E. Moise, G. Y. Rainich, R. L. Wilder. The Journal is published by the University of Michigan Press and will appear in lithoprinted form. Inasmuch as this necessitates close cooperation between author and typist, publication will in general be restricted to articles of authors living in or near Ann Arbor.

The Journal will appear semi-annually, two issues making one volume. Volume 1, No. 1, dated January, 1952, is now ready. The subscription price is $4.00 per year, with the special rate of $2.00 per year for individual research mathematicians. All inquiries and subscriptions should be addressed to Michigan Mathematical Journal, Mathematics Department, 3012 Angell Hall, University of Michigan, Ann Arbor, Michigan. Checks should be made payable to University of Michigan Press.

### PERSONAL ITEMS

Professor H. J. Ettlinger of the University of Texas was the representative of the Association at the inauguration of President J. W. Laurie of Trinity University on October 8, 1952.

Professor J. C. Polley of Wabash College represented the Association at the inauguration of President R. J. Humbert of DePauw University on October 18, 1952.

Professor H. W. Syer of Boston University was appointed to represent the

Association at the One Hundredth Anniversary of the Founding of Tufts College on October 11, 1952.

Professor A. E. Pitcher has been awarded a John Simon Guggenheim Memorial Foundation Fellowship and is on leave of absence from Lehigh University during the current academic year.

Duke University makes the following announcements: Associate Professor F. G. Dressel has been promoted to a professorship; Mr. W. L. Gordon and Mr. J. R. Shoenfield have been appointed to instructorships; Professor W. W. Rankin has retired with the title of Professor Emeritus.

At Illinois Institute of Technology: Associate Professor Haim Reingold has been appointed Acting Chairman of the Department of Mathematics; Dr. Allen Devinatz has been granted a leave of absence to accept a National Research Fellowship and will spend a year at the Institute for Advanced Study.

Michigan State College reports the following: Assistant Professor K. J. Arnold of the University of Wisconsin has been appointed to an associate professorship; Dr. Henry Parkus, recently consultant in aerodynamics with the National Pneumatic Company, Boston, has been appointed to an assistant professorship; Professor H. L. Harter has resigned to accept a position at the Wright-Patterson Air Force Base.

Montana State University announces: Professor A. S. Merrill, formerly chairman of the Department of Mathematics, has been appointed Dean of the Faculty and Dean of the College of Arts and Sciences; Professor Harold Chatland has been named Chairman of the Department; Dr. W. M. Meyers, Jr., formerly an assistant instructor at Ohio State University, has been appointed to an assistant professorship; during the 1952 Summer Session Professor Paul Reichelderfer of Ohio State University was a visiting professor in the Department.

Rutgers University makes the following announcements: Assistant Professor M. G. Galbraith has been promoted to an associate professorship; Teaching Assistant John Bender has been promoted to an instructorship; Dr. Solomon Leader, formerly a research assistant at Princeton University, Mr. B. H. McCandless, previously a teaching assistant at Indiana University, Mr. R. E. Montgomery who has been with the Atomic Energy Commission, Albuquerque, New Mexico, Dr. V. L. Shapiro, formerly a teaching assistant at the University of Chicago, Mr. H. G. Tucker, a teaching assistant at the University of California, Berkeley, and Dr. K. G. Wolfson, formerly a university fellow at the University of Illinois, have been appointed to instructorships; Dr. R. K. Brown, formerly a teaching assistant at the University, is now a mathematician with the United States Signal Corps, Belmar, New Jersey.

University of Illinois announces: Professor Reinhold Baer is on sabbatical leave during 1952–53 and is traveling in Europe; Assistant Professors E. J. Scott and H. E. Vaughan have been on sabbatical leaves during the first semester of 1952–53; Assistant Professors P. T. Bateman and Josephine Mitchell have received research grants from the National Science Foundation for 1952–53;

Assistant Professor M. K. Fort, Jr. has received a Ford Foundation grant for 1952–53 and is on leave of absence for the year; Dr. Arno Jaeger of Goettingen, Germany, who has been teaching recently in Ibadan, Africa, and Dr. Chung-Tao Yang, recently at Tulane University, have been appointed Research Associates during 1952–53.

University of Mississippi announces the following: With the help of a grant from the General Education Board, the Division of Arts and Sciences of the University of Mississippi is beginning a five-year program for the improvement of the undergraduate work in this division. One of the five points of the program is concerned with special attention to the superior student. The work will be distinctly supplementary to the regular routine program. To help implement this program in mathematics, the University announces the appointment of Professor Lester R. Ford as Professor of Mathematics for the second semester of the 1952–53 session.

At the University of New Hampshire: Dr. R. M. Conkling of the University of Florida has been appointed to an assistant professorship; Mr. S. B. Hobbs, Mr. R. E. Hux, Mr. A. R. Lamontagne, and Miss Elizabeth A. Stone have been appointed to part-time instructorships.

Wayne University reports the following: Dr. H. D. Huskey, formerly an assistant director at the Institute for Numerical Analysis, Los Angeles, California, has been appointed to a professorship; Associate Professor Yu Why Chen of the University of Oklahoma and Professor Casper Goffman, University of Oklahoma, have been appointed to associate professorships; Miss Winifred Burroughs has been appointed to an instructorship.

Mr. J. B. Bartoo of State University of Iowa has been appointed to an assistant professorship at Pennsylvania State College.

Assistant Professor W. N. Birchby of the California Institute of Technology has retired.

Mrs. Barbara B. Blair, formerly a part-time instructor at State University of Iowa, has been appointed to an instructorship at Smith College.

Professor A. H. Blue of Culver-Stockton College has been appointed to an associate professorship at Cornell College.

Mr. L. F. Boron, previously a geophysicist with the Navy Department, Washington, D. C., has been appointed to an assistant professorship at Norwich University.

Dr. H. D. Brunk who has been with the Sandia Corporation, Albuquerque, New Mexico, has been appointed to an associate professorship at the University of Missouri.

Assistant Professor V. W. Burrows of Northeastern State College, Oklahoma, has resigned to become Coordinator of Secondary Education of City Schools, Tahlequah, Oklahoma.

Mr. R. G. Buschman, formerly a part-time instructor at the University of Colorado, has been appointed to an instructorship at McNeese State College.

Mr. L. G. Campbell is now on active duty with the United States Air Force and has been assigned to an instructorship at the United States Naval Academy.

Mr. W. L. Carter of Western Illinois State College has accepted a position with the Department of Education, Guam, Mariana Islands.

Professor P. F. Cauffman, formerly Head of the Department of Mathematics of Maryland State Teachers College, Salisbury, has been appointed Chairman of the Department of Mathematics of State Teachers College, Shippensburg, Pennsylvania.

Assistant Professor R. S. Christian of Georgia Institute of Technology has been appointed to an assistant professorship in the Atlanta Division of the University of Georgia.

Mr. D. E. Coffey who has been a graduate student at Montana State University is now Head of the Department of Mathematics of Saline High School, Michigan.

Assistant Professor D. A. Darling is on leave of absence from the University of Michigan during 1952–53 and has a position as Visiting Assistant Professor in the Department of Mathematical Statistics, Columbia University.

Mr. B. C. DeLoach, previously a student at Alabama Polytechnic Institute, has been appointed to a teaching fellowship at Ohio State University.

Professor W. L. Duren, Jr. of Tulane University has been serving as Acting Program Director for Mathematics for the National Science Foundation.

Dr. M. P. Emerson, previously a graduate assistant at the University of Illinois, has been appointed to an assistant professorship at Harpur College of the State University of New York.

Mr. J. E. Forbes of Bradley University has been appointed to a graduate assistantship at Purdue University.

Mrs. Virginia Forbes who has been teaching at Community Unit High School, Heyworth, Illinois, has been appointed to an assistantship at Purdue University.

Mr. H. H. Fox, formerly an assistant at the University of Illinois, has accepted a position as Research Physicist with Mound Laboratory, Miamisburg, Ohio.

Assistant Professor Abraham Franck of Kansas State College has a position as Senior Mathematician with Engineering Research Associates, St. Paul, Minnesota.

Mr. E. T. Frankel who has been on the staff of the Health and Welfare Federation of Allegheny County, Pennsylvania, has accepted a position as Regional Research Analyst with the Bureau of Public Assistance, Federal Security Agency, New York City.

Mr. N. S. Free, previously a lecturer at the University of California, Berkeley, has been appointed to an assistant professorship at Rensselaer Polytechnic Institute.

Professor H. M. Gehman, chairman of the Department of Mathematics of

the University of Buffalo, has been serving as Acting Dean of the Graduate School of Arts and Sciences during the first semester of 1952–53.

Mr. H. H. Goode has been appointed Director of the Willow Run Research Center, University of Michigan, Ypsilanti, Michigan.

Dr. L. W. Green, previously an assistant at Yale University, has been appointed to an instructorship at Princeton University.

Professor Emeritus F. L. Griffin of Reed College has a position as Visiting Professor at Wesleyan University.

Assistant Professor Simon Gruenzweig of Lincoln University, Pennsylvania, has been appointed Professor and Head of the Department of Mathematics and Physics of Philander Smith College.

Dr. B. F. Hadnot has been appointed to an instructorship at Florida State University.

Mr. D. K. Hartman of the University of Minnesota has accepted a position with General Electric Company, Syracuse, New York.

Mr. J. R. Hatcher, formerly an instructor at Howard University and more recently a graduate student at Brown University, has been appointed to an assistant professorship at Fisk University.

Mr. A. T. Hind has been appointed to an associate professorship at Clemson College.

Mr. T. R. Horton, formerly an instructor at Bolles School, Jacksonville, Florida, has been appointed to a graduate assistantship at the University of Florida.

Mr. S. L. Hull who was a research participant at Oak Ridge National Laboratory has been appointed to an assistant professorship at The Citadel.

Mr. H. F. Hunter who has been with the Northrop Aircraft Company, Hawthorne, California, has accepted a position as a mathematician at the United States Naval Radiological Defense Laboratory, San Francisco, California.

Assistant Professor L. S. Laws of the University of Minnesota has been appointed to a graduate assistantship in the School of Education, Michigan State College.

Associate Professor Joseph Lehner of the University of Pennsylvania is on leave of absence during the current academic year and is engaged as Staff Member at the Los Alamos Laboratory, New Mexico.

Mr. Stanislaw Leja has accepted a position with the Ford Motor Company, Buffalo, New York.

Associate Professor Caroline A. Lester of New York State College for Teachers, Albany, has been promoted to a professorship.

Dr. Octave Levenspiel, formerly a research engineer at the University of California, has been appointed to an assistant professorship in the Department of Chemical Engineering, Oregon State College.

Assistant Professor P. T. Mielke of Wabash College has a position as Stress Analyst with Boeing Airplane Company, Seattle, Washington.

Mr. D. G. Miller, previously a graduate student at the University of Illinois,

has been appointed to an assistant professorship in the Department of Chemistry of the University of Louisville.

Dr. Morris Ostrofsky has resigned from his position as Chairman of the Department of Mathematics of Duquesne University to accept a position as Advisory Scientist to the Atomic Power Division of the Westinghouse Electric Corporation, Pittsburgh, Pennsylvania.

Professor W. R. Ransom of Tufts College has retired with the title of Professor Emeritus after fifty years of service.

Mr. O. M. Rasmussen of the University of Kansas has been appointed to an assistant professorship at the University of Denver.

Mr. J. G. Renno, Jr., formerly a graduate assistant at the University of Wisconsin, has been appointed to an instructorship at the Milwaukee Extension of the University.

Dr. E. K. Ritter, formerly of the Willow Run Aeronautical Research Center, Ypsilanti, Michigan, has accepted a position as Director of Computation and Ballistics, United States Naval Proving Ground, Dahlgren, Virginia.

Associate Professor G. G. Roberts of Berea College has been promoted to a professorship.

Mr. P. C. Rogers, previously a graduate assistant at the University of Maryland, has accepted a position as a mathematician with the United States Air Force, Washington, D. C.

Professor Helen G. Russell of Wellesley College has been named Chairman of the Department of Mathematics.

Sister Mary Esther, formerly of Mundelein College, is now at St. Vincent Convent, Petaluma, California.

Mr. W. L. Stamey of the University of Missouri has been appointed to an assistant professorship in the Atlantic Division of the University of Georgia.

Professor C. G. Stipe of the Michigan College of Mining and Technology was engaged during the summer of 1952 by the Calumet and Hecla Consolidated Copper Company, Michigan.

Dr. D. L. Thomsen, previously of the Jet Propulsion Laboratory, Pasadena, California, has been appointed to an assistant professorship at Pennsylvania State College.

Dr. Peter Thullen has accepted a position in the International Labor Office, Geneva, Switzerland.

Mr. R. F. Tidd of Canisius College has been promoted to an assistant professorship.

Mr. C. H. Tross, previously a graduate assistant at the University of Illinois, has a position as a mathematician with the Wright Air Development Center, Dayton, Ohio.

Dr. W. R. Van Voorhis of Fenn College has been appointed Visiting Professor of Engineering Administration at Case Institute of Technology; he is also a member of the Case Operations Research Group.

Mr. C. R. Wampole, formerly a student at St. John's University, has been

appointed to an instructorship at Eastern Military Academy, Cold Spring Harbor, New York.

Mr. Chih-Yi Wang who has been a teaching assistant at the University of Minnesota has been appointed to an associate professorship at Hampton Institute.

Dr. Albert Wolinsky of New York University has accepted a position as Assistant Physicist with Farrand Optical Company, New York City.

Associate Professor Emeritus E. F. A. Carey of Montana State University died on May 19, 1952. He was a charter member of the Association.

Professor Emeritus E. V. Huntington of Harvard University died on November 25, 1952. He was a charter member and third President of the Association.

Mr. D. F. Peterson of Utah State Agricultural College died in September, 1952.

Associate Professor J. H. Pitman of the Department of Mathematics and Astronomy, Swarthmore College, died on September 23, 1952.

Professor Emeritus T. R. Running of the University of Michigan died on October 10, 1952. He was a charter member of the Association.

Professor Emeritus W. J. Rusk of Grinnell College died on September 10, 1952. Professor Rusk was a charter member of the Association.

Professor Otto Szasz of the University of Cincinnati died in Switzerland on September 19, 1952.

Professor Marie J. Weiss of Newcomb College, Tulane University, died on August 19, 1952. She was a member of the Board of Governors and served as an associate editor of the MONTHLY during 1940–46.

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## THE MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### COMBINED MEMBERSHIP LIST

The Mathematical Association of America and the American Mathematical Society are planning hereafter to publish a *Combined Membership List* in place of the separate lists which have been issued in the past. The 1952 edition will be sent only to members of the American Mathematical Society.

Members of the Association who are not members of the Society and who wish to purchase copies of the list may do so at $2.00 per copy from the office of the American Mathematical Society, 80 Waterman Street, Providence 6, Rhode Island.

HARRY M. GEHMAN
*Secretary-Treasurer*

## NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 74 persons have been elected to membership by the Board of Governors on applications duly certified.

T. A. ABOUHALKAH, Bacc. in Math. (France) Grad. Student, Petroleum Engineering, University of Texas.

W. P. ANDERSON, Student, Macalester College.

M. A. ARREOLA, M.S. (Philippines) Instr., Mapua Institute of Technology, Manila, Philippines.

J. L. BAGG, M.S. (Illinois) Grad. Assistant, Michigan State College.

R. E. BAYLES, B.A. (Brown) Grad. Student, Harvard University.

J. S. BERGEN, B.A. (New Jersey S.T.C., Montclair) 626 Joralemon Street, Belleville, N. J.

C. R. BERNDTSON, B.A. (Bridgeport) Mathematician, Nuclear Development Associates, White Plains, N. Y.

R. H. BOYER, Student, Carnegie Institute of Technology.

E. R. BRENTZEL, B.A. (Minnesota) Cartographer, Army Map Service, Washington, D. C.

R. A. BROMAN, M.A. (Indiana) Chairman, Department of Mathematics, Mishawaka High School, Ind.

J. C. BROOKS, M.A. (Georgia) Asst. Professor, Georgia Institute of Technology.

AZELLE BROWN, M.A. (Columbia) Instr., Hofstra College.

S. H. COHN, M.A. (Toronto) Instr., Fournier Institute of Technology, Lemont, Ill.

C. C. CRELL, M.S. (Miami) Instr., Miami University, Oxford, Ohio.

L. J. DIXON, M.S. (Oklahoma A. & M.) Asst. Professor, Arkansas State College.

MARTHA E. EDWARDS, Ph.D. (North Carolina) Teacher, Campbell College, Buies Creek N. C.

F. M. ELLIS, M.Ed. (Pittsburgh) Acting Dean, Engineering School, Youngstown College.

L. A. ELROD, M.A. (Kansas City) Mathematician, White Sands Proving Ground, Las Cruces, N. M.

M. A. FAMIGLIETTI, M.A. (Oklahoma) Mathematician, Aberdeen Proving Ground, Md.

D. L. FRIED, Student, Rutgers University.

A. S. GREGORY, Computer, Sandia Corporation, Albuquerque, N. M.

W. R. GRUBB, B.A. (Buffalo) Grad. Student, University of Buffalo.

FRED GRUENBERGER, M.S. (Wisconsin) Computing Service Project Supervisor, University of Wisconsin.

O. G. HARROLD, JR., Ph.D. (Stanford) Professor, University of Tennessee.

J. W. HOLLINGSWORTH, M.S. (Wisconsin) Assistant in Computing, University of Wisconsin.

R. W. HUFF, Student, College of Wooster.

JACK INDRITZ, M.S. (Chicago) Instr., University of Minnesota.

J. A. JACOBS, M.Ed. (Duke) Registrar; Head, Department of Mathematics, Pembroke State College, N. C.

BROTHER CYPRIAN JOHN, Ph.D. (Catholic) Instr., Manhattan College.

H. J. JOHNSON, B.S. (Oklahoma A. & M.) Engineer, American Telephone and Telegraph Company, St. Louis, Mo.

C. E. JONES, M.S. (Michigan) Asso. Professor, Engineering, Tennessee Agricultural and Industrial State College.

R. H. JONES, JR., M.S. (Union C.) Head Engineer, Navy Department, Bureau of Ships, Washington, D. C.

S. T. KAO, Ph.D. (Catholic) Asst. Professor, St. Joseph's College, Albuquerque, N. M.

A. A. KARWATH, M.S. (Iowa S. C.) Instr., St. Ambrose College.

M. L. KEEDY, M.A. (Nebraska) Asst. Professor, Physics, North Dakota State College.

GEORGE KLEIN, Ph.D. (Chicago) Asst. Professor, Mount Holyoke College.

T. R. KNAPP, B.A. (Rochester) Grad. Student, University of Rochester.

L. C. LABOWITZ, Student, University of Maryland.

H. S. LEONARD, JR., B.S. (Michigan S. C.) Grad. Student, Harvard University.

OCTAVE LEVENSPIEL, Ph.D. (Oregon S. C.) Asst. Professor, Chemical Engineering, Oregon State College.

T. A. LOVE, Ph.D. (N.Y.U.) Professor and Head, Department of Mathematics, Tennessee Agricultural and Industrial State College.

E. G. MACE, Toolmaker, Chevrolet Division of General Motors, Tonawanda, N. Y.

B. J. MARKS, M.S. (Iowa S. C.) Research Assistant, Merrill Project, University of Illinois.

MRS. DELEPHARE B. MAYHALL, M.A. (Mississippi) Head, Department of Mathematics, Itawamba Junior College, Fulton, Miss.

J. G. MCGOUGH, Student, School of General Studies, Columbia University.

BROTHER J. G. MCKENNA, M.A. (Columbia) Dean, Iona College.

R. J. MERCER, B.A. (California) Ensign, United States Navy, San Francisco, Calif.

W. K. MOORE, Ph.D. (Kansas) Asst. Professor, Albion College.

MRS. VERA T. MORRIS, B.A. (DePauw) Instr., Purdue University.

LOIS M. NEFF, M.A. (Minnesota) Teacher, Rochester Senior High School, Minn.

LOIS NICKCHEN, Student, Marquette University.

R. C. NICKERSON, Student, Brown University.

REV. C. F. O'CALLAGHAN, B.A. (St. Bonaventure) Instr., Siena College.

R. F. PAVLEY, Research Technician, Wayne University.

G. A. PAXSON, Student, University of Michigan.

GILBERT PUENTE, B.S. (Arizona) Mathematician, White Sands Proving Grounds, Las Cruces, N. M.

LOIS J. ROPER, M.A. (Missouri) Instr., Trenton Junior College, Mo.

F. M. SANDS, Student, Gonzaga University.

J. H. SARTAIN, M.S. (Chicago) Edwin Shields Hewitt and Associates, Libertyville, Ill.

MARLOW SHOLANDER, Ph.D. (Brown) Asso. Professor, Washington University.

WILLIAM SHULEVITZ, Student, Wayne University.

SISTER ANNA CONCILIO O'NEILL, M.A. (Columbia) Professor, College of St. Elizabeth.

S. E. SKLAR, M.S. (N.Y.U.) Instr., Brooklyn Technical High School; Instr., Engineering Cooper Union.

D. G. STECHERT, M.S. (Cincinnati) Research Engineer, Gates Rubber Company, Denver, Colo.

J. E. STEINKRAUS, Student, Marquette University.

R. L. STERNBERG, Ph.D. (Northwestern) Mathematician, Laboratory for Electronics, Boston, Mass.

E. H. STONE, B.A. (Denver) Teacher, Smiley Junior High School, Denver, Colo.

LISTON TATUM, M.A. (Northwestern) Administrative Assistant, International Business Machines Corporation, New York, N. Y.

TUNG TSANG, M.S. (Minnesota) Grad. Student, University of Minnesota.

W. T. TUTTE, Ph.D. (Cambridge) Asst. Professor, University of Toronto.

FLORENT VENNE, St. Jacques, Quebec Province, Canada.

MARY E. WILSON, B.A. (Vassar) Grad. Student, University of Chicago.

F. L. WOLF, M.A. (Washington) Instr., Carleton College.

MRS. GERALDYNE P. ZIMMERMAN, M.A. (South Carolina S. C.) Asst. Professor, South Carolina State College.

## THE THIRTEENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The thirteenth annual William Lowell Putnam Mathematical Competition will be held on Monday, March 23, 1953. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to undergraduate students in universities and colleges of the United States and Canada who have not received a degree. The examination will consist of two parts of three hours each.

The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor L. E. Bush, 112 Albertus Magnus Hall, College of St. Thomas, St. Paul 1, Minnesota, by a postcard request. All applications must be filed with the Director not later than March 1, 1953. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application. Fewer than three from one college or university may compete as individuals.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to this rule may be made by the Director in cases where it would entail unusual inconvenience to a contestant. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $400, $300, $200, and $100, in the order of their rank. In addition, there will be prizes of $40, $30, $20 and $10 awarded to the members of these teams according to the rank of the team; a prize of $50 to each of the five highest contestants and a prize of $20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to several individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual $2000 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on the completion of the student's undergraduate work.

Reports on the twelve previous competitions and examinations will be found in this MONTHLY for May, 1938, 1939, 1940, 1941, 1942, October, 1946, August-September, 1947, December, 1948, August–September, 1949, 1950, 1951, and October 1952.

## THE OCTOBER MEETING OF THE MINNESOTA SECTION

The October meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Scholastica in Duluth, Minnesota on October 11, 1952. Sessions were held in the forenoon, at luncheon and in the afternoon. Sister M. Mercedes, Professor W. R. McEwen and Professor A. G. Swanson, Chairman of the Section, presided at the respective sessions.

Thirty-seven persons attended the meeting including the following twenty-four members of the Association:

H. M. Anderson, F. J. Arena, L. E. Bush, R. H. Cameron, H. D. Colson, K. L. Hankerson, Ruth E. Henning, J. S. Hill, R. T. John, W. C. Kalinowski, Karlis Kaufmanis, P. G. Kirmser, C. B. Lindquist, K. O. May, W. H. McBride, W. R. McEwen, Sister M. Mercedes, E. O. Nelson, J. C. Peterson, L. W. Sheridan, F. C. Smith, O. E. Stanaitis, A. G. Swanson, Takashi Terami.

By invitation of the Executive Committee, Mr. J. S. Hill, Actuary of the Minnesota Mutual Life Insurance Co., delivered an address at the morning session entitled "Changing Horizons in Certain Areas of Applied Mathematics." Abstract of this address follows:

Modern scientific development has introduced new areas of applied mathematics. The address did not consider these new areas so much as it did the changes in actual mathematical methods themselves being wrought by modern computational and data-handling equipment. The nature of modern computational equipment was reviewed. Specific applications of the equipment were reviewed and the general nature of the resulting changes in mathematical method were considered.

The traditional mathematical approach to life insurance reserves was reviewed and the changing concepts produced by the introduction of modern computational equipment was demonstrated.

It was concluded that professional mathematicians (1) must recognize the resurgence of the empirical approach to problem-solving, (2) must decide whether to leave the teaching emphasis on the classic analysis methods or to train more computation-minded students, and (3) may still retain one of the greatest satisfactions of the teaching profession, namely, the enriching experience of seeing young minds develop into useful contributors to our modern society.

The following short papers were presented:

1. *A problem in arrangements*, by Professor W. R. McEwen, University of Minnesota, Duluth Branch.

There are many problems in which one wishes to count a number of arrangements in which certain arrangements are excluded by the conditions of the problem. The *problème des ménages* is an example. By virtue of their definition, determinants are peculiarly adapted to this kind of counting. This was illustrated by solving a special case of the above problem and two others in one of which the exclusion was not systematic.

2. *An estimate of variations in amplitude of forced vibrations*, by Professor P. G. Kirmser, University of Minnesota.

In investigating the stability of a machine for testing metals in vibration, it was found necessary to estimate the variations in magnitude of steady state forced vibrations caused by slight hunting of the driving motor.

The system was idealized to

$$m\ddot{x} + \beta\dot{x} + kx = a \sin (\omega t + \gamma \sin pt)$$

which has the steady state solution

$$x = \sum_{n=-\infty}^{+\infty} A_n J_n(\gamma) \sin [(\omega + np)t + \phi_n].$$

Variations in amplitude are estimated by

$$M \cong \frac{|J_{-1}(\gamma)|\,A_{-1} + |J_{+1}(\gamma)|\,A_{+1}}{|J_0(\gamma)|\,A_0}$$

where $A_n$ is the amplitude of the steady state vibration for the system

$$m\ddot{x} + \beta\dot{x} + kx = a \sin(\omega + np)t.$$

3. *On the history of the problem of the isochrone*, by Professor F. J. Arena, North Dakota State College.

In this paper, the author discusses the circumstances which led Leibnitz to propose the problem of the isochrone and James Bernoulli's analytical solution of the problem.

4. *Remarks on convergence of certain series*, by Professor O. E. Stanaitis, St. Olaf College.

The series

$$\sum_{n=1}^{\infty} \frac{\sin n^2\theta \sin n\theta}{n^\alpha}, \quad \sum_{n=1}^{\infty} \frac{\cos n^2\theta \sin n\theta}{n^\alpha}, \quad \sum_{n=1}^{\infty} \frac{\sin n^2\theta \cos n\theta}{n^\alpha}, \quad \sum_{n=1}^{\infty} \frac{\cos n^2\theta \cos n\theta}{n^\alpha}$$

for $0<\alpha<1$ were considered. It was shown that the first two series converge uniformly in every interval of real values of $\theta$. However, the third and fourth series are divergent.

5. *Three pathological functions*, by Mr. D. C. McGarvey, Carleton College, introduced by Professor K. O. May.

Three functions discontinuous on an everywhere dense set of points are presented which can be handled in an elementary and intuitively satisfying manner that lends them to classroom discussion. For rational $x$, $f_1(x)$ is the reciprocal of the decimal place in which $x$ begins the first repeating cycle, $f_2(x)$ is the reciprocal of the length of the cycle and $f_3(x)$ is the reciprocal of the decimal place in which $x$ begins the second cycle. For irrational $x$, all three functions are 0. All three functions are discontinuous for a rational $r$ since arbitrarily close to $r$ is an irrational number at which the functions are 0 yet at $r$ they are positive. At any irrational point $i$, $f_1$ is proved discontinuous by constructing a number arbitrarily close to $i$ at which $f_1=1$. Another number arbitrarily close to $i$ can be constructed such that $f_2=1$ so that $f_2$ is also everywhere discontinuous. That $f_3$ is continuous on the irrationals is established by proving that in an interval there are but a finite number of points at which $f_3>\epsilon$ so that a $\delta$ can always be found excluding these points.

6. *Direct currents in infinitely long parallel wires*, by Professor P. C. Rosenbloom and Mr. T. T. Wu, University of Minnesota.

Let $D_1, D_2, \cdots, D_n$ be simply connected bounded domains with disjoint closures in the $xy$-plane, and let $D_0$ be the exterior of the union. We suppose the boundaries to be smooth. We imagine erected on $D_1, \cdots, D_n$ cylinders parallel to the $z$-axis representing conductors of given magnetic permeabilities $\mu_i$ and conductivities, and we assume that the exterior is a medium of permeability $\mu_0$ and conductivity 0. We suppose that currents are running parallel to the $z$-axis in the conductors, that the system is in steady state, and that the total current in each cross section $D_i$ is given. We set up integral equations for determining the field produced. The necessary and sufficient condition for the magnetic energy in each cross section to be finite is that the total current be zero. We compute the inductance coefficients and show that for non-ferromagnetic materials Maxwell's formulae in terms of the geometrical mean distance are correct to within errors of the order $10^{-6}$,

$$\lambda_i - 1 = 10^{-6}.$$

F. C. SMITH, *Secretary*

## CALENDAR OF FUTURE MEETINGS

Thirty-fourth Summer Meeting, Queen's University and the Royal Military College, Kingston, Ontario, August 31–September 1, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA, Ball State Teachers College, Muncie, May 2, 1953.

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS

KENTUCKY, University of Louisville, Spring, 1953.

LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Spring, 1953.

MICHIGAN, Western Michigan College of Education, Kalamazoo, April 11, 1953.

MINNESOTA, St. Olaf College, Northfield, May 9, 1953.

MISSOURI, William Jewell College, Liberty, Spring, 1953.

NEBRASKA

NORTHERN CALIFORNIA, San Francisco State College, January 31, 1953.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, Montana State University, Missoula, June 19, 1953.

PHILADELPHIA

ROCKY MOUNTAIN, University of Colorado, Boulder, April, 1953.

SOUTHEASTERN, Alabama Polytechnic Institute, Auburn, March 13–14, 1953.

SOUTHERN CALIFORNIA, Los Angeles City College, March 14, 1953.

SOUTHWESTERN

TEXAS

UPPER NEW YORK STATE, United States Military Academy, West Point, Spring, 1953.

WISCONSIN, Mount Mary College, Milwaukee, May, 1953.

## THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9 and 10, additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. These numbers have been issued to date:

No. 1. *Calculus of Variations* by G. A. Bliss, xii+189 pages.

No. 2. *Analytic Functions of a Complex Variable* by D. R. Curtiss, ix+173 pages.

No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.

No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.

No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.

No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.

No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.

No. 8. *Rings and Ideals* by N. H. McCoy, xii +216 pages.

No. 9. *The Theory of Algebraic Numbers* by Harry Pollard, xii+143 pages.

No. 10. *The Arithmetic Theory of Quadratic Forms* by B. W. Jones, x+212 pages.

# ELEMENTARY THEORY OF EQUATIONS

Samuel Borofsky, Brooklyn College

In addition to presenting a course in theory of equations, this book helps to bridge the gap between classical and so-called "modern algebra." It acquaints the student with some facts concerning the roots of algebraic equations and methods for obtaining them and at the same time introduces him to some of the concepts of present-day algebra. Also included are excellent problem lists with both mechanical problems and others that are instructive and challenging.

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John F. Randolph, University of Rochester

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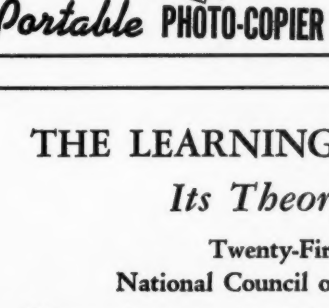

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*275 Pages with tables; 199 Pages without tables 6¼" × 9¼" Published 1952*

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## FLUID DYNAMICS. Fourth Symposium on Applied Mathematics

M. H. MARTIN, Editor, University of Maryland. In press

This book consists of papers presented at the University of Maryland meeting in June 1951 of the American Mathematical Society. Contributors include C. C. Lin, G. F. Carrier, A. Weinstein, S. Chandrasekhar, L. Onsager, G. Birkhoff, J. L. Synge and many others.

## STATISTICAL QUALITY CONTROL. New 2nd Edition

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## INTRODUCTION TO THE THEORY OF GAMES

By J. C. C. MCKINSEY, Stanford University. 382 pages, $6.50

Deals with the mathematical theory of games of strategy. The mathematical apparatus developed by the author will find application not only in the pure form of parlor games of strategy (chess, bridge, poker, etc.), but also in economics, statistics, and the theory of military strategy.

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330 WEST 42ND STREET, NEW YORK 36, N. Y.


GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

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WASHINGTON D.C.

# THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 60 NUMBER 2

PUBLIC LIBRARY
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